M306 Cosmolgy Notes
Based on the 2014 spring lectures by
Dr C Boehmer.

17/1/14

Books:
S. Dadelson
Modern Cosmology
Academic Press.
—/—
A Liddl and D. Lyth
Cosomological Inflation and Large Scale structure
—/—
S. Carrollo
Spacetime and geometery
Addiseon Wesley.
Lectures notes on GR gr-qc 9712019
—/—
J. Plebanski and A. Krasinski
Intro to GR and Cosmology
CUP.

1 Cosmological models.

1.1 Preliminaries.

Recall the Einstein field equations:

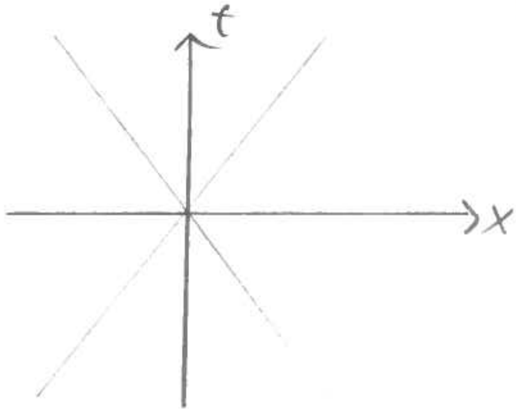
$$G_{ab} = 8\pi T_{ab} \left[\frac{8\pi G}{c^4} \right]$$

where G_{ab} is the Einstein tensor

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$$

where R_{ab} is the Ricci tensor. and R is the

Ricci scalar. g_{ab} is the metric tensor of a 4-dimensional Lorentzian space-time.



T_{ab} is the energy-momentum tensor. The field equation express the interaction between matter and geometery.

Wheeler: spacetime tells matter how to move and matter tells spacetime how to curve.

The key question of classical cosmology is which solution of the Einstein field equations describe the (idealised) evolution of the Universe that we observe.

In order to make some qualitative statements about the Universe, we need the following three axioms:

<u>Axiom 1</u>: The laws of physics are valid everywhere and at all time.

<u>Axiom 2</u>: The fundamental constants c, G, h, k_B, ... are true constants (in particular

no time dependence.

Axiom 3: The Universe is connected.

1.2 Introduction.
Cosmological observations show that the number of galaxies in any volume of size of about 100 Mpc is roughly the same. Hence the universe looks uniform on these scales.

$$1\,pc = 3.26\ ly.$$

Also, the isotropic distribution of radio galaxies, observed red shift in the spectra of distant galaxies, and the cosmic microwave background radiation (CMB) are almost constant for varying angles of observations.

Therefore, we assume that the universe is the same everywhere over large enough scales and thus we assume the universe to be homogenous and isotropic. This, along with Axiom 1 is often called the cosmological principle.

Definition 1.1 Homogeneity. A system is homogeneous if it is invariant under translations $x^a \to x^a + A^a$.

<u>Def</u> 1.2 : Isotropy. A system is isotropic if it is invariant under rotations.

We are interested in a special set of observers who see an isotropic universe. This defines a special reference frame.

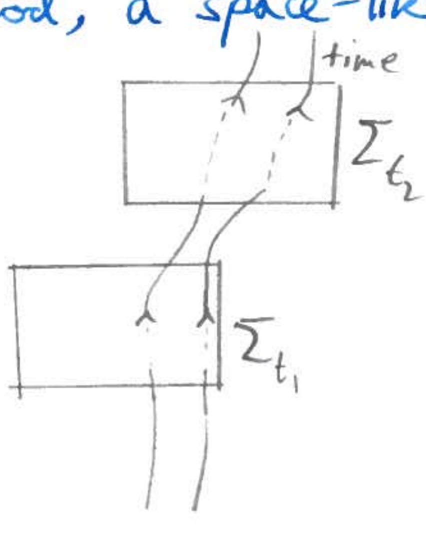
<u>Def</u>: 1.3 : Cosmological reference frame. A set of coordinates in which physical quatities are homogenous and isotropic.

<u>Def</u> 1.4 : Cosmological observer. An observer at rest in the cosmological reference frame.

<u>Def</u> 1.5 : Cosmic time : The proper time t measured by a comoving observe, starting with $t=0$ at the big bang.

1.3 The Friedmann - Lemaître - Robertson - Walker (FLRW) metric.

A cosmic time t, the universe define a 3-dimensional manifold, a space-like hypersurface



Homogeneity implies that the Universe expands uniformly. Also, the rate of expansion is the same everywhere.

Homogeneity also implies that the constant time hypersurfaces are spaces of constant curvature. Any two such spaces of the same dimension and the same signature are locally isometric

Spaces of vanishing curvature are flat Euclidean space, spaces of constant positive curvature are spheres, and spaces of constant negative curvature are hyperboloids.

The metric ansatz for a locally homogeneous and isotropic metric is:

$$ds^2 = -dt^2 + a^2(t) \frac{dx^2 + dy^2 + dz^2}{\left(1 + \frac{k}{4}(x^2+y^2+z^2)\right)}$$

$a(t)$ is called the scale factor.
k is a constant. Our metric signature is $(-,+,+,+)$.

—/—

$$ds^2 = -N^2(t)\, dt^2 + a^2(t) \ldots$$

$$N^2(t)\, dt^2 = d\tau^2$$
$$N(t)\, dt = d\tau$$
$$\Rightarrow \tau = \int N(t)\, dt.$$

—/—

Let us denote the spatical part of the meteric as:

$$\gamma_{ij}\, dx^i dx^j$$

such that

$$ds^2 = -dt^2 + a^2(t)\, \gamma_{ij}\, dx^i dx^j$$

where x^i $i = 1, 2, 3$ are the spatial coordinates.

The Ricci scalar of the spatial part of the meteric is:

$$^{(3)}R = \frac{6k}{a^2}$$

please check

In 3 dimenscion the Ricci scalar is directly related to the intrinic curvature of the, maniflod. Therefore, the sign of k dermines the sign of the spatial Ricci tensor and hence fixes the geometery.

However, the complete Ricci sclar is:

$$R = \frac{6k}{a^2} + \frac{\dot{a}^2}{a^2} + 6\frac{\ddot{a}}{a}$$

$$\ldots = {}^{(3)}R + 6\frac{\dot{a}^2}{a^2} + 6\frac{\ddot{a}}{a}.$$

Its sign is not determined by k.
R is the time dependent.

$$S_{GR}' = \int R \sqrt{-g}\, d^4x$$

1.4 Geometry of the constant time hypersurfaces
We will now discuss the spatial geometeries in more detail and introduce various different coordinates.

We will work with the three cases $k = \{1, 0, -1\}$.

— / — (Not in lecture notes)

$$ds^2 = -dt^2 + a^2(t) \frac{dx^2 + dy^2 + dz^2}{\left(1 + \frac{7k}{4}(x^2+y^2+z^2)\right)^2}$$

$$\left(1 + \frac{k}{4}\underbrace{(\sqrt{7}x)}_{\tilde{x}}{}^2 + \underbrace{(\sqrt{7}y)}_{\tilde{y}}{}^2 + \underbrace{(\sqrt{7}z)}_{\tilde{z}}{}^2\right)^2$$

$$\begin{aligned} d\tilde{x} &= \sqrt{7}\,dx \\ d\tilde{y} &= \sqrt{7}\,dy \\ d\tilde{z} &= \sqrt{7}\,dz \end{aligned} \quad \Rightarrow \quad d\tilde{x}^2 = 7dx^2$$

etc.

$$\Rightarrow ds^2 = -dt^2 + a^2(t) \frac{\frac{1}{7}(d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2)}{\left(1 + \frac{k}{4}(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)\right)^2} \quad \text{is}$$

$$= -dt^2 + \tilde{a}^2(t) \frac{(d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2)}{\left(1 + \frac{k}{4}(\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2)\right)^2}$$

—/—

Change of coordinates.
Let us introduce spherical polar coordinates.

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta .$$

which gives

$$x^2 + y^2 + z^2 = r^2$$

$$dx = \sin\theta \cos\phi \, dr + r\cos\theta \cos\phi \, d\theta$$
$$- r\sin\theta \sin\phi \, d\phi .$$

By chain rule $dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta$

$+ \ldots$

$$dx^2 = \sin^2\theta \cos^2\phi \, dr^2 + r^2\cos^2\theta \cos^2\phi \, d\theta^2$$
$$+ r^2\sin^2\theta \sin^2\phi \, d\phi^2$$
$$+ 2r\sin\theta \cos\theta \cos^2\phi \, dr \, d\theta$$

$$-2r\sin^2\theta\sin\phi\cos\phi\,dr\,d\phi$$

$$-2r^2\cos\theta\sin\theta\cos\phi\sin\phi\,d\theta\,d\phi.$$

$$\Rightarrow \underbrace{\gamma_{ij}\,dX^i dX^j}_{\frac{dx^2+dy^2+dz^2}{\left(1+\frac{k}{4}(\dots)\right)}} = \frac{dr^2+r^2d\Omega^2}{\left(1+\frac{k}{4}r^2\right)^2}$$

—/—

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$dx = \cos\theta\,dr - r\sin\theta\,d\theta$$
$$dx^2 = \cos^2\theta\,dr^2 - 2r\sin\theta\cos\theta\,dr\,d\theta + r^2\sin^2\theta\,d\theta^2$$

$$dy = \sin\theta\,dr + r\cos\theta\,d\theta$$
$$dy^2 = \sin^2\theta\,dr^2 + 2r\sin\theta\cos\theta\,dr\,d\theta + r^2\cos^2\theta\,d\theta^2.$$

$$\Rightarrow ds^2 = dx^2 + dy^2$$
$$= dr^2 + r^2\,d\theta^2.$$

—/—

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$.

Next, one introduces a new radial coordinate

$$\rho = \frac{r}{\left(1 + \frac{k}{2} r^2\right)}$$

$$\Rightarrow \rho^2 = \frac{r^2}{\left(1 + \frac{k}{4} r^2\right)^2} \Rightarrow \frac{r^2 d\Omega^2}{\left(1 + \frac{k}{4} r^2\right)^2} = \rho^2 d\Omega^2$$

The result is

$$\frac{dr^2}{\left(1 + \frac{k}{4} r^2\right)^2} = \frac{d\rho^2}{1 - k\rho^2}$$

$$\Leftarrow d\rho = \left(\frac{\left(\left(1 + \frac{k}{4} r^2\right) - \left(\frac{k}{4} 2r\right)\right)}{\left(1 + \frac{k}{4} r^2\right)}\right) dr$$

$$= \frac{1 - \frac{k}{4} r^2}{\left(1 + \frac{k}{4} r^2\right)^2} dr$$

—/—

$$\left[ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi\right]$$

$$ds^2 = dy^2 + dz^2 \quad z = \frac{1}{x}$$

$$\frac{dx^2}{x^4} + dy^2$$

—/—

$$d\rho^2 = \frac{\left(1 - \frac{k}{4} r^2\right)^2 dr^2}{\left(1 + \frac{k}{4} r^2\right)^4}$$

$$\frac{1}{1+k\rho^2} = \frac{1}{1 - k\left(\frac{r}{(1+\frac{k}{4}r^2)}\right)^2} = \frac{1}{1 - \frac{kr^2}{(1+\frac{k}{4}r^2)^2}}$$

$$= \frac{1}{\frac{\left(1+\frac{k}{4}r^2\right)^2 - kr}{\left(1+\frac{k}{4}r^2\right)^2}}$$

$$= \frac{\left(1+\frac{k}{4}r^2\right)}{1 + \frac{k}{2}r^2 + \frac{k^2}{16}r^4 - kr}$$

$$= \frac{\left(1+\frac{k}{4}r^2\right)^2}{\left(1-\frac{k}{4}r^2\right)^2}$$

$$\Rightarrow \frac{d\rho^2}{1-k\rho^2} = \frac{\left(1-\frac{k}{4}r^2\right)^2}{\left(1+\frac{k}{4}r^2\right)^4} \cdot \frac{\left(1+\frac{k}{4}r^2\right)^2}{\left(1-\frac{k}{4}r\right)^2} \, dr^2$$

The final is:

$$\gamma_{ij}\, d\chi^i d\chi^j = \frac{d\rho^2}{1-k\rho^2} + \rho^2 d\Omega.$$

This is the most common form of the FLRW metric.

Constant positive curvature $k=+1$.
We have:

$$\gamma_{ij}\, d\chi^i d\chi^j = \frac{d\rho^2}{1-\rho^2} + \rho^2 d\Omega^2.$$

$$\int \frac{d\rho}{\sqrt{1-\rho^2}}$$ ← from

$$\rho = \sin\chi$$
$$d\rho = \cos\chi \, d\chi$$
$$d\rho^2 = \cos^2\chi \, d\chi^2$$

$$\frac{d\rho^2}{1-\rho^2} = d\chi^2$$

$$\Rightarrow \gamma_{ij}\, d\chi^i d\chi^j = d\chi^2 + \sin^2\chi \, d\Omega^2$$

$$= d\chi^2 + \sin^2\chi \, d\theta^2 +$$

$$\sin^2\chi \sin^2\theta \, d\phi^2.$$

This is the light element of the 3-sphere (S^3) Every $\chi = \chi_0$ = const. hypersurface [or $\theta = \theta_0$ or $\phi = \phi_0$] corresponds to a 2-sphere. (cuts through 3 spheres gives 2-spheres.

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$$\gamma_{ij}\, dX^i dX^j = \frac{d\rho^2}{1-k\rho^2} + \rho^2 d\Omega^2$$

$$d\chi^2 = \frac{d\rho^2}{1-k\rho^2}$$

$$d\chi = \frac{d\rho}{\sqrt{1-k\rho^2}}$$

$$\chi = \int \frac{d\rho}{\sqrt{1-k\rho^2}}$$

curvature

$k = +1$

$$\gamma_{ij}\, dX^i dX^j = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

To see this metric indeed describe S^3, we consider the space $\mathbb{R}^4$ with coordinates (x, y, z, w) and

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2$$

and choose

$$x = \sin\chi \sin\theta \cos\varphi .$$
$$y = \sin\chi \sin\theta \sin\varphi .$$
$$z = \sin\chi \cos\theta$$
$$w = \cos\chi .$$

One verifies:

$$x^2 + y^2 + z^2 + w^2 = 1.$$

How do we find ds^2 in the new coordinates?

$$dx = \cos\chi \sin\theta \cos\phi \, d\chi$$
$$+ \sin\chi \cos\theta \cos\phi \, d\theta$$
$$- \sin\chi \sin\theta \sin\phi \, d\phi .$$

$dx^2 = \ldots$ and compute all other terms. We arrive at

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 + dw^2 \\ &= d\chi^2 + \sin\chi \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \end{aligned}$$

and have shown that this the metric of the 3-sphere S^3.

Lets us consider the volume of S^3.

$$V = \int \sqrt{\det \gamma_{ij}} \, d^3x$$

$$= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sin^2\chi \sin\theta \, d\chi \, d\theta \, d\phi .$$

$$= 2\pi^2 .$$

<u>NB</u>:

$$\begin{pmatrix} 1 & & \\ & \sin^2\chi & \\ & & \sin^2\chi \sin^2\theta \end{pmatrix} \rightarrow \det = \sin^4\chi \cdot \sin^2\theta$$

Constant negative curvature $k = -1$

$$d\chi = \int \frac{d\rho^2}{\sqrt{1-\rho^2}}$$

$$\Rightarrow \rho = \sinh \chi$$

$$d\rho = \cosh \chi \; d\chi$$

$$d\rho^2 = \cosh \chi \; d\chi^2$$

$$\frac{d\rho^2}{1+\rho^2} = \frac{\cosh^2 \chi}{1+\sinh^2 \chi} d\chi^2 = d\chi^2 .$$

Therefore, we can write the metric of the so-called hyperspace H^3 as follows:

$$\gamma_{ij} \, dX^i dX^j = \frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega^2$$

$$= d\chi^2$$

$$+ \sinh^2 \chi \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right)$$

The volume of this space is unbounded.

Zero curvature $k = 0$

$$\gamma_{ij} \, dX^i dX^j = d\rho^2 + \rho^2 d\Omega^2 .$$

which is flat space in spherical polars

Summary:
We can write the FLRW metric in the following form:

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \Sigma(\chi)^2 d\Omega^2\right]$$

where:

$$\Sigma(\chi) = \begin{cases} \sin\chi & \text{if } k = +1 \\ \chi & \text{if } k = 0 \\ \sinh\chi & \text{if } k = -1. \end{cases}$$

It is imporant to know what is ment by distance If we work with ρ as our radial coordinate

$$\left[\frac{d\rho^2}{1-k\rho^2}\right]$$

then we have to be careful when measuring distance if $k = \pm 1$. In this we have to integrate:

$$\frac{d\rho}{\sqrt{1-k\rho^2}}.$$

1.5 Particle horizons.

When analysing cosmological models the following question naturally arises: How much of our universe can be observed in principle at a given event p.

In cosmology, the test particles and observers are assumed to be galaxies. Hence, we wish to know which (isotropic) observers could have sent a signal which reaches another observers at or before p.

The boundary between world lines that can reach p and those that cannot is called the particle horizon at p. Let us assume a universe of finite age and consider the flat FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right]$$

$$= -a^2(\tau)\, d\tau^2 + a^2(\tau)\left[dx^2 + dy^2 + dz^2\right]$$

$$= a^2(\tau)\left[-d\tau^2 + dx^2 + dy^2 + dz^2\right]$$

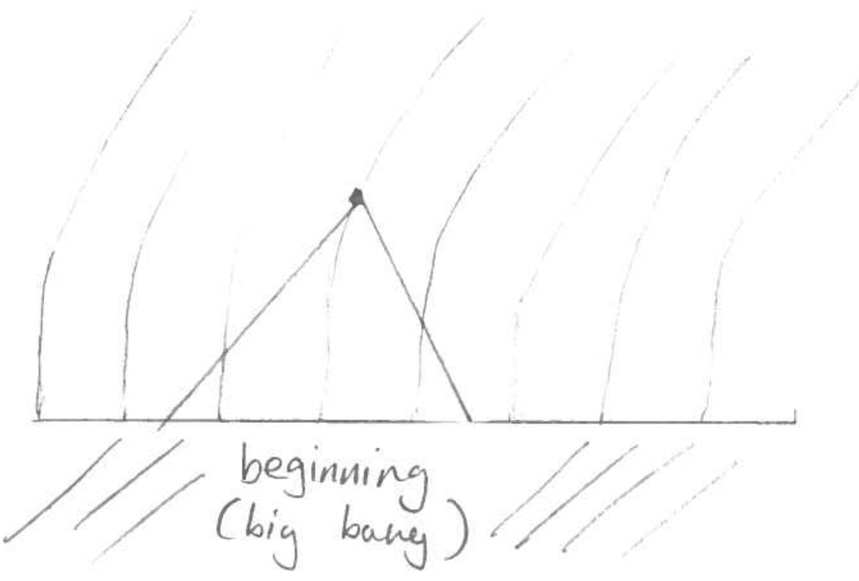
$$-a^2(\tau)\, d\tau^2 = -dt^2$$
$$\Leftrightarrow a(\tau)\, d\tau = dt$$
$$\Leftrightarrow d\tau = \frac{dt}{a(t)}$$

$$t = \int a(\tau)\, d\tau \,, \quad \boxed{\tau = \int \frac{dt}{a(t)}}$$

t is called cosmological time,
τ is conformal time.

This is a multiple of the Minkowski metric and so all coordinate range from $-\infty$ to $+\infty$.

Let us assume the universe began at $t=0$ and consider an observer at p. If τ diverages as $t \to 0$ the observer will be able to receive signals from all other observers. The intergal diverges if $a(t) \leq \alpha t$ for $\alpha = \text{const}$ as $t \to 0$ and there will be no particle horizon since τ will range down to $-\infty$. If, on the other hand, the integral converages, there exists a particle horizon because only a portion of τ is covered.



1.6 Matter, content and field equatioi.

Background:

We must now think the matter part of the Einstein equations. The matter is describe by the energy-momentum tensor T_{ab}.

We will use the ideal gas perfect fluid:

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b)$$

ρ ... energy density.
P ... pressure
u_a ... fluid 4-velocity which satisfies $g_{ab} u^a u^b = -1$.

—/—

count equations:

2 field equations

unknowns

$$a(t),\ \rho(t),\ P(t)$$

$\Rightarrow$ need more information.

—/—

We need to prescribe an equation of state to close the system of equations.

We choose (for simplicity) an equation of state of the form

$$P = w\rho.$$

where $w = \text{const}$.

Standard values of w are

$$w = \begin{cases} -1 & \text{dark energy.} \\ 0 & \text{matter, dust; matter-dominated} \\ 1/3 & \text{radiation, radiation dominated} \\ 1 & \text{stiff matter} \end{cases}$$

$w < -1$ phantom cosmologies.
$w > 1$ super stiff models.

Field equations:
It is customary to insert the cosmological constant Λ (dark energy) on the left-hand side of the field equations

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}.$$

At this point we would start with the FLRW metric, compute Christoffel symbols, the Riemann tensor, the Ricci tensor and eventually the Einstein field equations.

The Ricci tensor components are

$$R^t_t = \frac{3\ddot{a}}{a}.$$

$$R^x_x = R^y_y = R^z_z = \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2}.$$

$$R^a_b = 0 \text{ if } a \neq b.$$

$$R = 6\frac{\ddot{a}}{a} + \frac{6\dot{a}^2}{a^2} + \frac{6k}{a^2}.$$

where the dot means d/dt with respect to cosmological time. Then, the Einstein tensor components are

$$G^t_t = -3\frac{\dot{a}^2}{a^2} - 3\frac{k}{a^2}.$$

$$G^x_x = G^y_y = G^z_z = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2}.$$

The energy-momentum tensor is

$$T^a_b = \mathrm{diag}(-\rho, P, P, P)$$

Thus, the field equations are:

$$(tt) \quad -3\frac{\dot{a}^2}{a^2} - 3\frac{k}{a^2} + \Lambda = -8\pi\rho.$$

$$(\text{spatial}) \quad -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} + \Lambda = 8\pi P.$$

We rewrite this a bit:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$-2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = 8\pi P - \Lambda + \frac{k}{a^2}$$

We add both equations and get:

$$-2\frac{\ddot{a}}{a} = \frac{8\pi}{3}(\rho + 3P) - \frac{2}{3}\Lambda$$

$$\Leftrightarrow \quad -\frac{\ddot{a}}{a} = \frac{4\pi}{3}(\rho + 3P) - \frac{\Lambda}{3}$$

$$\Leftrightarrow \quad \frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}.$$

We collect two equations

$$\boxed{\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3} \end{aligned}}.$$

These are the so called Friedmann equations which are the starting point of analysing concreteeing solutions.

<u>Continuity equation</u>.

The twice contracted Bianchi identities imply

$$\nabla_a T^{ab} = 0.$$

These equations are not independent. We can

Kin Quan.

MATH 3306 Cosmology Problem sheet 3

Please hand in your solutions Friday, 28th February 2014

Problem 1 (40 points) Consider the cosmological field equations with cosmological constant, $k = 1$ and a perfect fluid with equation of state $P = w\rho$. Now start with

$$a(t) = a_0 + \varepsilon\delta a_1(t) \tag{1}$$

$$\rho(t) = \rho_0 + \varepsilon\delta\rho_1(t) \tag{2}$$

where a_0 and ρ_0 are constants. (i) Determine the values of a_0 and ρ_0 for $\varepsilon = 0$, the background equations. (ii) Next, put the perturbations into the field equations and assume $\varepsilon \ll 1$ (Taylor expand the field equations, keeping lowest order terms only). (iii) Show that

$$\delta\ddot{a}_1 \propto (1 + w)(1 + 3w)\delta a_1. \tag{3}$$

(iv) Are perturbations stable for radiation?

Problem 2 (20 points) Show that in a flat cosmological model

$$\sigma(r_1) = r_1 = \int_t^{t_1} \frac{dt'}{a(t')}. \tag{4}$$

(i) Use the definition of the Hubble parameter to change the integration variable from t' to a, assume $a(t) = a$ and $a(t_1) = 1$. (ii) Integrate the resulting equation in a matter dominated universe with $k = \Lambda = 0$.

Problem 3 (40 points) Show that the definition of the density parameter and the cosmological field equations imply the following relation

$$\Omega = \Omega_1 \frac{(1 + z)}{1 + z\Omega_1} \tag{5}$$

where $\Omega = \Omega(t)$ etc. Furthermore show

$$H^2 = H_1^2(1 + z)^n(1 + z\Omega_1). \tag{6}$$

What is the value of n?

Xerox WorkCentre 5845
SMTP Transfer Report

Job Status: FAILED Job canceled by user.

Job Information

Device Name:	M0113MA3
Submission Date:	16/01/14
Submission Time:	17:41
Images Scanned:	12
Size:	0
Attachment Name:	
Format:	Image-Only PDF
Encrypted E-mail:	No

SMTP Server

Address:	144.82.111.57:25

Message Settings:

Subject:	Scanned from a Xerox Multifunction ...
From:	kin.quan.10@ucl.ac.uk
Reply To:	kin.quan.10@ucl.ac.uk
To:	

1. kin.quan.10@ucl.ac.uk

see this explicity by working with the field equations.

—/—

(1) $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}$$ (first).

(2) $$-2\frac{\ddot{a}}{a} - \left(\frac{\ddot{a}}{a}\right)^2 = 8\pi P - \Lambda + \frac{k}{a^2}$$

—/—

Dot the first eqn (1):

$$2\left(\frac{\dot{a}}{a}\right)\left[\frac{\ddot{a}a - \dot{a}^2}{a^2}\right] = \frac{8\pi}{3}\dot{\rho} + \frac{2k}{a^2}\frac{\dot{a}}{a}.$$

3(1) + (2):

$$8\pi(\rho + P) - \frac{2k}{a^2} = -2\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2$$

$$\Rightarrow 8\pi(\rho + P)\frac{\dot{a}}{a} = \underline{\frac{\dot{a}}{a}\frac{2k}{a^2}} + \underbrace{2\left[-\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a}\right]\frac{\dot{a}}{a}}_{= -\frac{8\pi}{3}\dot{\rho} - \underline{\frac{2k}{a^2}\frac{\dot{a}}{a}}}.$$

$$\Rightarrow \frac{8\pi}{3}\left\{\dot{\rho} + \frac{3\dot{a}}{a}(\rho + P)\right\} = 0.$$

This is the energy - momentum conservation equation of cosmology

—/—

$$\dot{\rho} + \frac{3\dot{a}}{a}(1+\omega)\rho = 0 .$$

$$\dot{\rho}/\rho = -3(1+\omega)\frac{\dot{a}}{a} .$$

$$(\log \rho)^{\cdot} = -3(1+\omega)(\log a)^{\cdot}$$

$$\log \rho = -3(1+\omega)\log a + C_1 .$$

$$\rho \propto a^{-3(1+\omega)} .$$

—/—

1.7 Cosmological solutions of the field equations

So far, we have derived the equations and now we want to study specific solutions. We will do this using different forms of matter.

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1.7.1 Friedman solution - matter dominated universe.

A matter dominated is described by the energy-momentum tensor.

$$T_{ab} = \rho u_a u_b \quad [P = 0].$$

If we neglect the cosmological constant for now, the Einstein field equations are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2}.$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\rho.$$

and we have the conservation which is:

$$\dot{\rho} + \frac{3\dot{a}}{a}\rho = 0$$

We can integrate the conservation equation

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}.$$

$$(\ln\rho)^{\cdot} = -3(\ln a)^{\cdot}$$

$$\Rightarrow \quad \ln \rho = -3 \ln a + C.$$

$$\Rightarrow \quad \rho = \rho_0 \frac{1}{a^3} \qquad \boxed{\rho \propto \frac{1}{a^3}}$$

Then the other field equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} \rho_0 \frac{1}{a^3} - \frac{k}{a^2}$$

zero spatial curvature $- k = 0$.

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi\rho_0}{3}} \frac{1}{a^{3/2}}$$

$$\frac{da}{dt} = \sqrt{\frac{8\pi\rho_0}{3}} a^{-1/2}$$

$$a^{1/2} da = \sqrt{\frac{8\pi\rho}{3}} dt.$$

$$\frac{2}{3} a^{3/2} = \sqrt{\frac{8\pi\rho_0}{3}} (t - t_0)$$

$$a^{3/2} = \sqrt{6\pi\rho_0} (t - t_0)$$

$$\Rightarrow a(t) = (6\pi\rho_0)^{1/3} (t - t_0)^{2/3}$$

Lets us assume that the universe started at time $t = t_0$ with "size" $a(t_0) = 0$

$$\Rightarrow a(t) = (6\pi\rho_0)^{1/3}\, t^{2/3}$$

$$\rho(t) = \rho_0 / a(t)^3 = \rho_0 / 6\pi\rho_0 \cdot t^{-2}.$$

$$\Rightarrow \left\{ \begin{matrix} \text{matter dominated} \\ \text{universe} \\ k = 0 \end{matrix} \right. , \quad \left(\begin{matrix} a(t) \propto t^{2/3} \\ \rho(t) \propto t^{-2} \\ \rho \propto \frac{1}{a^3} \end{matrix} , \quad \frac{\dot{a}}{a} = \frac{2}{3t} \right)$$

The energy density decrease as a function of time, however, it diverages as $t \to 0$. This corresponds to the big bang. Note, that a diveragent energy-density implies divergent curvature. These singularities seem to be "natural" in cosmological models.

These exists a particles horizon at $r = 3t/a$.

Positive spatial curvature :- $k = +1$.

We have the same conservation and get :

$$\rho = \rho_0 / a \cdot$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho_0}{3} \cdot \frac{1}{a^3} - \frac{1}{a^2}.$$

Separation of variables gives:

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi\rho_0}{3} \cdot \frac{1}{a^3} - \frac{1}{a^2}}$$

$$= \sqrt{\frac{1}{a^2}\left[\frac{8\pi\rho_0}{3}\frac{1}{a} - 1\right]}$$

$$\frac{da}{dt} = \sqrt{\frac{8\pi}{3}\rho_0 \frac{1}{a} - 1}$$

$$dt = \frac{da}{\sqrt{\frac{8\pi}{3}\rho_0\frac{1}{a} - 1}}.$$

$$a = b^2 \quad , \quad da = 2b\, db \quad .$$

$$\Rightarrow dt = \frac{2b\,db}{\sqrt{\frac{8\pi}{3}\rho_0\frac{1}{b^2} - 1}}$$

$$= \frac{2b\, db}{\frac{1}{b}\sqrt{\frac{8\pi}{3}\rho_0 - b^2}}.$$

$$t = \frac{8\pi}{3}\rho_0 \arctan\left(\frac{1}{\sqrt{\frac{8\pi}{3}\rho_0\frac{1}{a} - 1}}\right)$$

$$-\sqrt{a} \cdot \sqrt{\frac{8\pi}{3}\rho_0 - a}$$

We cannot solve this explicitly for $a(t)$. However, we can write this in a neat parametrised form!

$$\frac{1}{\sqrt{\frac{8\pi\rho_0}{3}\frac{1}{a}-1}} = \tan u.$$

$$\frac{1}{\frac{8\pi\rho_0}{3}\frac{1}{a}-1} = \tan^2 u.$$

$$\frac{8\pi}{3}\rho_0 = \frac{1}{\tan^2 u} + 1 = \frac{1}{\sin^2 u}$$

$$\Rightarrow a = \frac{8\pi}{3}\rho_0 \sin^2 u.$$

Therefore we get

$$t = \frac{8\pi}{3}\rho_0 \left(u - \sin u \cos u\right)$$

Then
$$\sin^2 u = \frac{1}{2}\left(1-\cos(2u)\right)$$

$$\sin u \cos u = \frac{1}{2}\sin(2u).$$

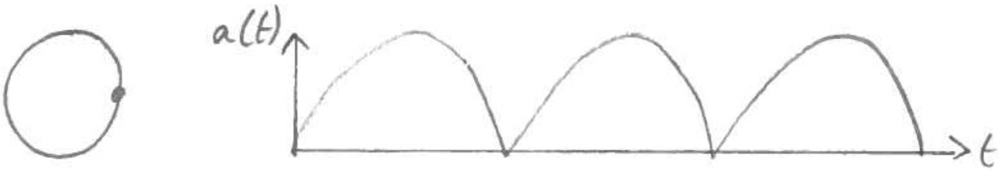
$$\Rightarrow a = \frac{8\pi}{3}\rho_0 \left(1-\cos(2u)\right)$$

$$t = \frac{8\pi}{3}\rho_0\left(\frac{2u}{2} - \frac{1}{2}\sin(2u)\right)$$

finally rescaling $u \to u/2$ $\quad (v = 2u)$

$$\Rightarrow a = \frac{4\pi\rho_0}{3}(1 - \cos v)$$

$$t = \frac{4\pi}{3}\rho_0(v - \sin v)$$



This is the standard parametrisation of the cycloid. At $t=0$ we have $v=0$ which gives $a=0$.

This universe expands until it reaches a maximal values.

$$a_{max} = \frac{4\pi\rho_0}{3} \cdot 2 = \frac{8\pi}{3}\rho_0$$

when $v = \pi$. This corresponds to:

$$t_{max} = \frac{4\pi}{3}\rho_0(\pi - 0) = \frac{4\pi^2}{3}\rho_0.$$

After t_{max}, this universe starts contracting.

until it reaches $a = 0$, at $t = 8\pi^2\rho_0/3$. This is often called the big cruch.

Negative spatial curvature $- k = -1$

In this case, we need to solve.

$$dt = \frac{da}{\sqrt{\frac{8\pi\rho_0}{3} \cdot \frac{1}{a} + 1}}$$

If we do a long calculation similar to previous, we arrive at:

$$a = \frac{4\pi}{3}\rho_0 \left(\cosh(2u) - 1\right)$$

$$t = \frac{4\pi}{3}\rho_0 \left(\sinh(2u) - 2u\right)$$

This expands as $t \to \infty$, asymptotically like $a(t) \sim t$

Assume $u \gg 1$ $\sin(2u) - 2u$.

$$= \frac{1}{2}\left(e^{2u} - e^{-2u}\right) - 2u.$$

$$\sim \frac{1}{2} e^{2u}$$

$$\cos(2u) - 1 = \frac{1}{2}\left(e^{2u} + e^{-2u}\right) - 1$$

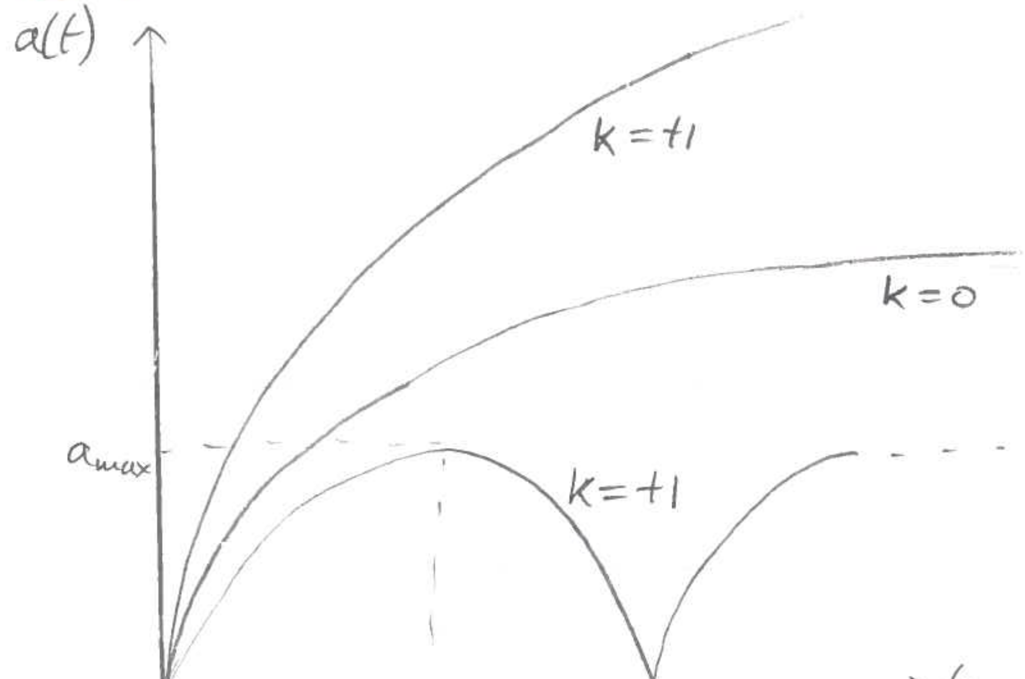
$$\sim \frac{1}{2} e^{2u}.$$

$$\Rightarrow a(u) \sim \frac{4\pi}{3}\rho_0 \frac{1}{2} e^{2u}$$

$$t(u) \sim \frac{4\pi}{3}\rho_0 \cdot \frac{1}{2} e^{2u}$$

$$\left.\begin{matrix} a(u) \\ t(u) \end{matrix}\right\} a \sim t.$$

Let us summerise these three results in a graph.



$$\rho \propto \frac{1}{a^3}$$

1.7.2 Friedman solutions - radiation dominated universe

A sufficiently hot and dense universe can no longer be describe by a matter dominated model. Radiation becomes the main source of the gravitational field and is described by the equation of state.

$$P = \frac{1}{3}\rho .$$

Let's integrate the conservation equation

$$\dot{\rho} + \frac{\dot{a}}{a}(\rho + P) = 0.$$

$$\dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{1}{3}\rho\right) = 0.$$

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0.$$

$$(\ln\rho)^{\cdot} = -4(\ln a)^{\cdot}$$

$$\Rightarrow \rho = \frac{\rho_0}{a^4} \qquad \left(\rho \propto \frac{1}{a^4}\right)$$

We now consider the spatially flat $(k=0)$ radiation dominated universe.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_0 \frac{1}{a^4}$$

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3}\rho_0}\, \frac{1}{a^2}$$

$$\frac{da}{dt} = \sqrt{\frac{8\pi}{3}\rho_0}\, \frac{1}{a}.$$

$$a\, da = \sqrt{\frac{8\pi\rho_0}{3}}\, dt$$

$$\frac{1}{2}a^2 = \sqrt{\frac{8\pi\rho_0}{3}}\,(t - t_0)$$

$$\Rightarrow a(t) = \sqrt{2}\left(\frac{8\pi\rho_0}{3}\right)^{1/4}(t-t_0)^{1/2}$$

Let us choose $t_0 = 0$ so that $a(0) = 0$, then

$$a(t) = \sqrt{2}\left(\frac{8\pi}{3}\rho_0\right)^{1/4}\sqrt{t}.$$

$$\boxed{a(t) \propto \sqrt{t}}$$

$$\frac{\dot{a}}{a} = \frac{1}{2t}$$

(Exam: Slove for $k=+1$)

$$\rho(t) \propto \frac{1}{a^4} \propto \frac{1}{t^2}$$

The particle horizon is located at $r = 2t/a$

1.8 Freidman - Lemaître solutions

So far, we have neglected the cosmological constant. Next, we will inculde this term in the field equations.

1.8.1 The Einstein static universe.

We consider the field equation with $\Lambda \neq 0$. Can we find a static solution?

$$a(t) = a_E = \text{const}$$

$$\rho(t) = \rho_E = \text{const}.$$

$$P(t) = 0.$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}.$$

$$\Rightarrow \frac{8\pi}{3}\rho_E + \frac{\Lambda}{3} - \frac{k}{a_E^2} = 0.$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}$$

$$\Rightarrow 0 = -\frac{4\pi}{3}\rho_E + \frac{\Lambda}{3} \Rightarrow \Lambda = 4\pi\rho_E$$

and therefore $\frac{8\pi}{3}\rho_E + \frac{4\pi\rho_E}{3} = \frac{k}{a_E^2}$.

$$\Rightarrow 4\pi \rho_E = \frac{k}{a_E^2}.$$

$$\Rightarrow k \neq 0$$
$$\& \ k \neq -1.$$

$$\Rightarrow k = +1.$$

Thus the Einstein is characterised by the following conditions.

$$\Lambda = 4\pi \rho_E.$$

and $\frac{1}{a_E^2} = 4\pi \rho.$

$$\Leftrightarrow a_E = \frac{1}{\sqrt{4\pi\rho_E}} = \frac{1}{\sqrt{\Lambda}}$$

These solution is unstable with respect to small perturbations $\rho \to \rho_0 + \delta\rho$ and $a \to a_0 + \delta a$.

1.8.2 The de Sitter solution.

Let us try to solve the field equations in the absense of matter $\rho = p = 0$ but with Λ

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3}$$

$$\Rightarrow \frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}} \Leftrightarrow (\ln a)^{\cdot} = \sqrt{\frac{\Lambda}{3}}$$

$$\ln a = \sqrt{\frac{\Lambda}{3}}\, t + C.$$

$$\Rightarrow a(t) = a_0 e^{\sqrt{\frac{\Lambda}{3}} t}.$$

with $a(t=0) = a_0$. The line element of this solution is given by:

$$ds^2 = -dt^2 + a_0^2 e^{2\sqrt{\frac{\Lambda}{3}} t} (dx^2 + dy^2 + dz^2)$$

It seems that the de Sitter metric is time-depenant, it turns out, it is static. Consider the time translation $t \to t' + t_0$.

Then
$$ds'^2 = -dt'^2 + a_0^2 e^{2\sqrt{\frac{\Lambda}{3}} t'} e^{2\sqrt{\frac{\Lambda}{3}} t} [dx^2 + dy^2 + dz^2]$$

Next, we introduce

$$x' = e^{\sqrt{\frac{\Lambda}{3}} t_0} x$$
$$y' = \ldots$$
$$z' = \ldots$$

$$\Rightarrow ds^2 = -dt'^2 + a_0^2 e^{2\sqrt{\Lambda/3}\,t'}\left[dx^2 + dy^2 + dz^2\right].$$

which is the original line element with primes.

Thus, the metric is invarient under time - translations and thus static.

1.8.3 Qualitative analysis of Friedman - Lemaître solutions.

Recall that in a matter dominated universe we have:

$$\rho \sim \frac{1}{a^3} \Rightarrow K_m = \frac{8\pi}{3}\rho a^3 = \text{const}.$$

Similarly, when radiation dominates we have

$$\rho \propto \frac{1}{a^4} \Rightarrow K_r = \frac{8\pi}{3}\rho a^4 = \text{const}.$$

Let us write $\rho_{total} = \rho_m + \rho_r$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho_{total} + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$\dot{a}^2 = \frac{8\pi}{3}\rho_m a^2 + \frac{8\pi}{3}\rho_r a^2 + \frac{\Lambda}{3}a^2 - k.$$

$$= \frac{K_m}{a} + \frac{K_r}{a^2} + \frac{\Lambda}{3}a^2 - k.$$

$$\Leftrightarrow \underbrace{\dot{a}^2}_{E_{kin}} \underbrace{- \frac{K_m}{a} - \frac{K_r}{a^2} - \frac{\Lambda}{3}a^2}_{E_{pot}} = -k.$$

$$E_{kin} + E_{pot} = const$$

which is of the form of a 1d mechanical system

$$\dot{a}^2 + V_{eff}(a) = -k.$$

with

$$V_{eff}(a) = -\frac{K_m}{a} - \frac{K_r}{a^2} - \frac{\Lambda}{3}a^2$$

This equation describes the motion of our idealised universe.

Without radiation and without Λ, we have.

$$\dot{a}^2 - \frac{K_m}{a} = -k.$$

$$\frac{M}{2}\dot{a}^2 - \frac{M}{2}\frac{1}{a}\left(\frac{8\pi}{3}\rho_m a^3\right) = -\frac{M}{2}k.$$

$$\frac{M}{2}\dot{a}^2 - \frac{GM^2}{a} = \text{const}.$$

Don't worry about the constant

This last equation we can read as kinetic energy + potential energy = const. It looks exactly like the Newtonian equation

—/—

$$G_{ab} = \kappa T_{ab}$$

—/—

7/2/14.

Without radiation and cosmological constant we find

$$\frac{M}{2}\left(\frac{da}{dt}\right)^2 - \frac{GM^2}{a} = \text{const} = -\frac{kM}{2}$$

where we used $\boxed{K_m = \frac{8\pi}{3}\rho a^3}$

$$= 2\left(\frac{4\pi}{3}a^3\rho_m\right) = 2M.$$

We can read this:

"Kinetic energy" + "Potential energy" = constant.

—/—

$\nabla_a T^{ab} = 0.$

$$\Rightarrow \dot{\rho}_{total} + \frac{3\dot{a}}{a}(\rho_{total} + P_{total}) = 0.$$

$$\rho_{total} = \rho_m + \rho_n + \ldots$$

The field equation imply conservation of the total matter, but not individual matter components

$$\dot{\rho}_m + \frac{3\dot{a}}{a}(\rho_m + P_m) + Q$$

matter $+\dot{\rho}_r + \frac{3\dot{a}}{a}(\rho_r + P_r) - Q = 0$

radiation

We assume that every matter component is independently conserved.

In the analogous Newtonian 2-body problem we have

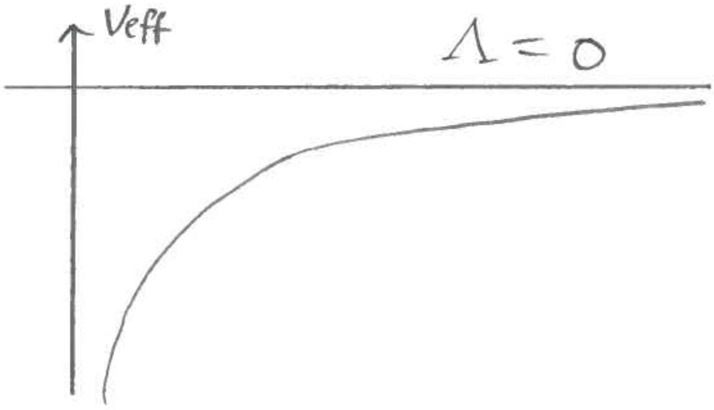
$$\frac{m\dot{r}^2}{2} + V_{eff}(r) = \text{const}.$$

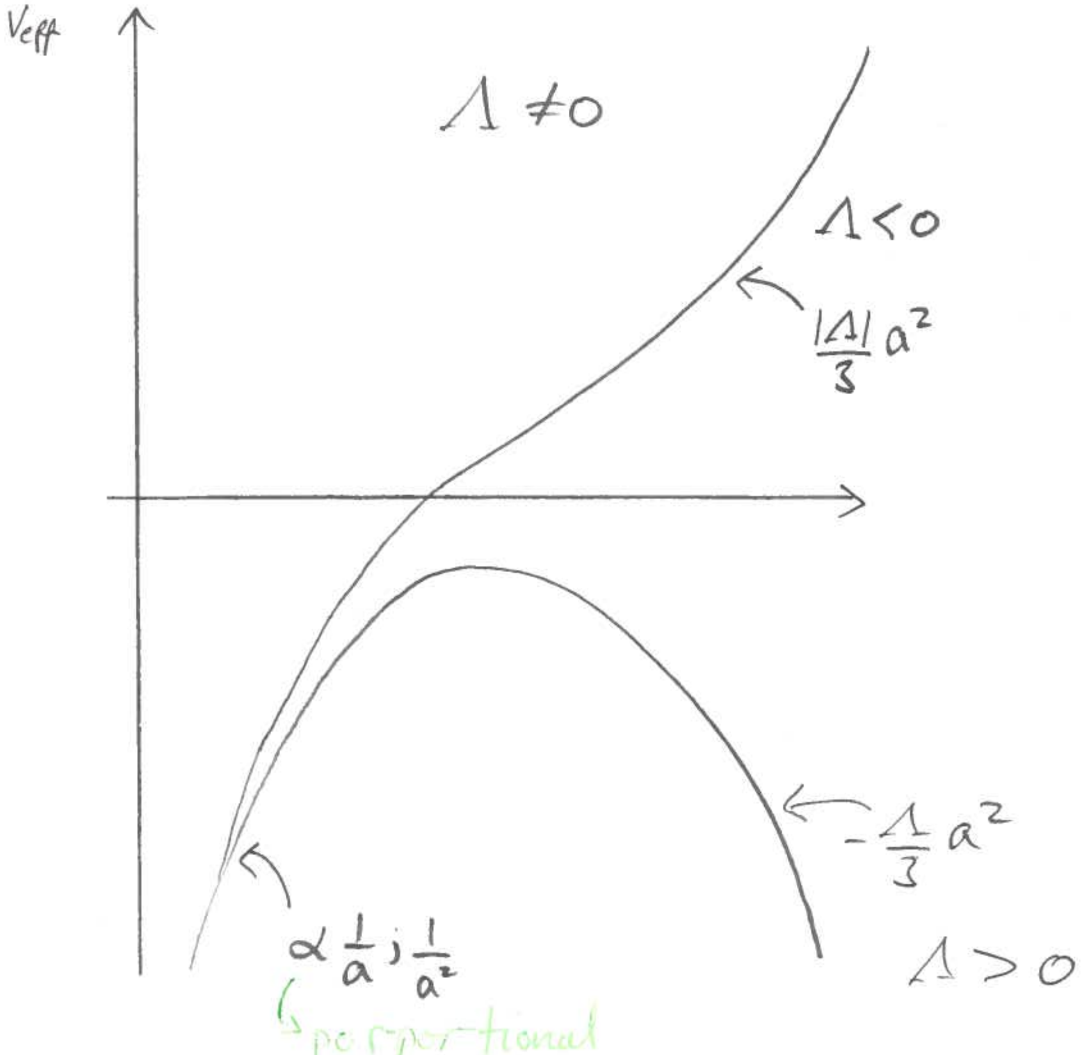
Therefore we can conclude that $k=1$ to closed orbits (ellipses), $k=-1$ to unbound orbits (hyperbola) and $k=0$ corresponds to the limiting case. Let us apply this to cosmology

$k=1$ universe expands and later contracts.
$k=0$ expansion velocity approches zero.
$k=-1$ kinetic energy energy dominates, expantion never stops].

[this only applies to $\Lambda = 0$]

$$V_{eff} = -\frac{k_r}{a^2} - \frac{k_m}{a} - \frac{\Lambda}{3} a^2 .$$





2 Cosmological parameters and observable quantities.

2.1 Cosmological parameters

In order to fix the expanseon parameter $a(t)$ in the FLRW meteric we would required an impossibly large number of observatcons. We can however approximate it with a finite number of observation.

Hubble parameter

The Hubble parameter gives the rate of expanseon

$$\boxed{H = \frac{\dot{a}}{a}}$$

The value of this parameter in the present epoch is the source of a great deal of contro--versy.

Measurements fall in the range of:

$$40 \frac{km}{sec\ Mpc} \leqslant H_0 \leq 90 \frac{km}{sec\ Mpc}$$

$$H_0 = 70 \frac{km}{sec Mpc}$$

$$Mpc - 10^6 pc = 10^6 \times 10^{24}\ cm$$

Note that we have to divide $\dot{a}$ by a as the vaule of $a(t_{today})$ is arbitrary.

<u>Deceleration parameter</u>

Next, we have the deceleration parameter which the rate of change of the expansion.

$$q_0 = \frac{-a\ddot{a}}{\dot{a}^2}$$

$$= -\frac{1}{H^2}\frac{\ddot{a}}{a}$$

The n-th parameter is given by:

$$q_n = (-1)^{n+1} \frac{a^{(n+2)}}{H^{n+2} a}$$

$$a^{(n+2)} = \frac{d^{n+2}}{dt^{n+2}} a$$

In physical units we have

$$H = \frac{1}{a}\frac{da}{dt}, \quad [H] = \frac{1}{\text{time}} \qquad [H/c] = \frac{1}{\text{length}}$$

The q_n is dimensionless.

Given a fixed time t_0, the quantities $H(t_0)$ and $q_n(t_0)$ are in principle observations. Hence, we approximate $a(t)$ by

$$a(t) = a(t_0)\left[1 + x - \frac{q_0}{2} x^2 + \frac{q_1}{3!} x^3 + \ldots\right]$$

where $x = (t - t_0) H_0$.

$$a(t) = a(t_0) + a(t_0)(t - t_0) H_0$$
$$- a(t_0)\frac{q_0}{2}(t - t_0)^2 H_0^2 + \ldots$$

$$\ldots = a(t_0) + a(t_0)\frac{\dot{a}(t_0)}{a(t)}(t-t_0)$$

$$-a(t_0)\frac{1}{2}(-1)\left(\frac{1}{H_0^2}\frac{\ddot{a}(t_0)}{a(t_0)}\right)(t-t_0)^2 H_0^2 + \ldots$$

$$\ldots = a(t_0) + \dot{a}(t_0)(t-t_0) + \frac{1}{2}\ddot{a}(t_0)(t-t_0)^2 + \ldots$$

H is the Hubble function, often called the Hubble const in which case it refers to today's value
H measures the relative expansion rate.

Density parameters.
Another useful quantity is the density parameter.

Let us start with the first field equation.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$H^2 = \frac{8\pi}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}$$

Let us divide this equation by H^2.

$$1 = \frac{8\pi\rho}{3H^2} + \frac{\Lambda}{3H^2} - \frac{k}{a^2H^2}$$

Thus, each term on the right is dimensionless.
It is useful to introduce some dimensionless parameter:

$$\Omega = \frac{8\pi\rho}{3H^2} = \frac{\rho}{\rho_c}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$

where ρ_c is the critical density define by:

$$\rho_c = \frac{3H^2}{8\pi}$$

which is a function of time. This results in the following form of the field equations

$$\Leftrightarrow 1 = \Omega + \Omega_\Lambda - \frac{k}{a^2H^2}$$

If we now introduce

$$\Omega_{total} = \Omega + \Omega_\Lambda$$

then we find

$$\boxed{\frac{k}{a^2H^2} = \Omega_{total} - 1}$$

In general Ω_{total} is a time dependant function. However, if $k = 0$ then $\Omega_{total} = 1$ at all times. Note that is equation in principle allows to determine the sign of k by measuring the total matter content of the universe.

What we are seeing is a link between geometry and matter coming from the very principle of GR.

Example: Consider matter dominated universe with $\Lambda = 0$ $(P = 0)$ $[k = 0]$

$$\rho \propto \frac{1}{a^3}, \quad a \propto t^{2/3}, \quad H = \frac{\frac{2}{3}t^{-1/3}}{t^{2/3}} = \frac{2}{3}\frac{1}{t}.$$

$$q_0 = -\frac{1}{H^2}\frac{\ddot{a}}{a} = -\frac{1}{\left(\frac{2}{3}\frac{1}{t}\right)^2} \cdot \frac{\frac{2}{3}\left(-\frac{1}{3}\right)t^{-4/3}}{t^{2/3}}$$

$$= \frac{2/9}{4/9 \cdot 1/t^2 \cdot t^{6/3}} = \frac{1}{2}.$$

$$\Omega = \frac{8\pi\rho}{3H^2} = \frac{8\pi\rho_0 \cdot 1/a^3}{3H^2} = \frac{8\pi\rho_0 \cdot \frac{1}{a_0^3 t^2}}{3\left(\frac{2}{3}\frac{1}{t}\right)^2}$$

$$= \frac{8\pi\rho_0 \cdot \frac{1}{a_0^3}}{4/3} \overset{!}{=} 1 \left(\frac{8\pi\rho_0}{4/3 \cdot a_0^3}\right)$$

Redshifts.

Electromagnetic singals are described by null geodesics. Start with the element in the form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\Omega^2\right].$$

Let us consider radial geodesics $\theta = \theta_0, \phi = \phi_0$.

Then $ds^2 = 0$.

$$-dt^2 + \frac{d\rho^2}{1-k\rho^2} = 0.$$

$$\Leftrightarrow \quad dt^2 = \frac{a^2(t)\, d\rho^2}{1-k\rho^2}$$

$$\Leftrightarrow \quad dt = \frac{\pm\, a(t)\, d\rho}{\sqrt{1-k\rho^2}}.$$

$$\Leftrightarrow \quad \frac{dt}{a(t)} = \pm \frac{d\rho}{\sqrt{1-k\rho^2}}$$

$$\Rightarrow \pm \int_{t_0}^{t_1} \frac{dt'}{a(t')} = \int_{r_1}^{r_0} \frac{d\rho}{\sqrt{1-k\rho^2}}$$

The limits and the signs of these integrals depend on where we define our coordinate orgin. We choose the observer to be at the centre, this means $r_0 = 0$. Note, time will be increaseng from r_1 to r_0, thus $t_1 < t_0$. This situatoin corresponds to:

$$\int_0^{r_1} \frac{d\rho}{1-k\rho^2} = -\int_{t_0}^{t_1} \frac{dt'}{a(t')}$$

We define

$$\sigma(r) = \int_0^r \frac{d\rho}{\sqrt{1-k^2\rho}} \geq 0 .$$

Let us assume a second signal is sent at $t_1 + \delta t_1$ which reaches us at $t_0 + \delta t_0$. We find

$$-\int_{t_0+\delta t_0}^{t_1+\delta t_1} \frac{dt'}{a(t')} = \sigma(r_1)$$

$$\Rightarrow \sigma(r_0) = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt'}{a(t')}$$

We take the difference of these two integral ($\int_{t_0}^{t_1}$ and $\int_{t_0+\delta t_0}^{t_1+\delta t_1}$) to see if something has changed

$$\underbrace{\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt'}{a(t')}}_{} - \int_{t_1}^{t_0} \frac{dt'}{a(t')} = 0 .$$

$$\int_{t_0}^{t_0+\delta t_0} + \cancel{\int_{t_1}^{t_0}} - \int_{t_1}^{t_1+\delta t_1} - \cancel{\int_{t_1}^{t_0}} \frac{dt'}{a(t')}$$

$$\Leftrightarrow \int_{t_0}^{t_0+\delta t_0} \frac{dt'}{a(t')} - \int_{t_1}^{t_1+\delta t_1} \frac{dt'}{a(t')} = 0 .$$

$$H(\delta t) = \int_{t_0}^{t_1+\delta t} f(s)\, ds$$

$$H(\delta t) = H(0) + H'(0)\,\delta t + \frac{H''(0)}{2!}(\delta t)^2 + \dots$$

$$\Rightarrow \int_{t_0}^{t_0} \frac{dt'}{a(t)} + \frac{1}{a(t_0)}\delta t_0 - \int_{t_1}^{t_1} - \frac{1}{a(t_1)}\delta t_1$$
$$+ \mathcal{O}(\delta t_0^2, \delta t_1^2)$$

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)}$$

We can interpert δt_0 and δt_1 as the time intervals between maxima of the signal's wave
Then we may write:

$$\delta t_0 = \lambda_0 \qquad \delta t_1 = \lambda_1$$

and the redshift is defined by

$$z_1 = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\lambda_0}{\lambda_1} - 1 = \frac{a(t_0)}{a(t_1)} - 1.$$

$$\Rightarrow \boxed{\frac{a(t_0)}{a(t_1)} = 1 + z}.$$

14/02/14

$$1+z_1 = \frac{a(t_0)}{a(t_1)}$$ (Imporant) [Past paper may have different notation]

2.3 Luminosity distance.

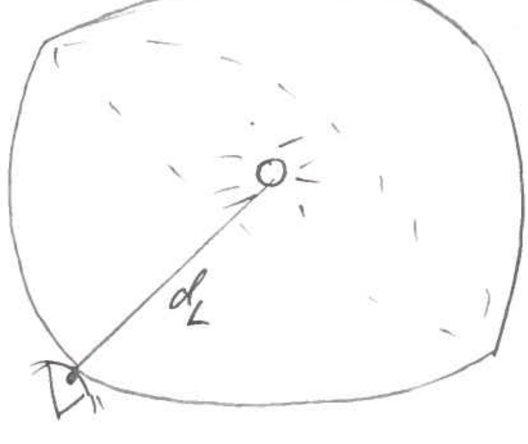
First, we recall that power P = energy / unit time. We have two types of types of luminosity.

- intrinsic quantities
  - absolute luminosity L. total power radiated by the object.

- observed quantities
  - apparent luminosity l the power received per unit area by the telescope.

We start by using the Euclidean definition as our definition of luminosity distance. The light/radiation of an object expands in spherical wavefronts. Thus, at distance d_L from the object, we can relate the apparent and absolute luminosities.

$$L = 4\pi d_L^2 l.$$



We can solve this for d_L and use this as a definition of luminosity distance.

$$d_L = \sqrt{\frac{L}{4\pi \ell}}$$

Next, we need to understand how this changes in a cosmological setting.

Consider an object with redshift z_1, which emitted light at t_1 and has radical coordinate r_1. Let us try to express d_L. We will change coordinate so that the spherical wavefronts are the centres. Thus, if we as the observer are at $r = r_1$ and the object is at $r = 0$ the line element becomes (at our position).

$$d\tau^2 = a^2(t)\left[r_1^2 \, d\Omega^2\right]$$

To see this write the FLRW metric in coordinates (t, r, θ, ϕ). This line element describes a sphere with effective radius $a^2(t) r_1^2$ and effective radius $a(t) r_1$, and effective area $4\pi(a(t)r_1)^2$.

Now, we will observe this means $t = t_0$,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

and we write $a(t_0) = a_0$ and the light/radiation has spread over the area.

$$A = 4\pi a_0^2 r_1^2 .$$

Now, let $L_{received}$ be the total power received at the sphere with radius $a_0 r_1$. Then

$$l = \frac{L_{received}}{4\pi^2 a_0 r_1^2}.$$

The received luminosity at our time t_0 may not be the same as the absolute luminosity at t_1 redshift decrease the energy of light, and, as will see, time dilation slows its delivery.

The power reaching the sphere at radius $a_0 r_1$ is

$$L_{received} = \frac{\Delta E_0}{\Delta t_0}.$$

while $L_{emitted} = \Delta E_1 / \Delta t_1$

(a) redshift : For a photon of frequency ν and energy $h\nu$.

$$\frac{\nu_0}{\nu_1} = \frac{1}{(1+z_1)}.$$

Therefore we have:

$$\frac{E_0}{E_1} = \frac{1}{(1+z_1)}$$

(b) time dilation; from above we have

$$\frac{\Delta t_2}{\Delta t_1} = (1+z_1)$$

Therefore $$\frac{L_{emitted}}{L_{received}} = \frac{\Delta E_1}{\Delta t_1} \cdot \frac{\Delta t_0}{\Delta E_0}.$$

$$= \frac{\Delta E_1}{\Delta E_0} \cdot \frac{\Delta t_0}{\Delta t_1}$$

$$= (1+z_1)(1+z_1)$$

$$= (1+z_1)^2.$$

and

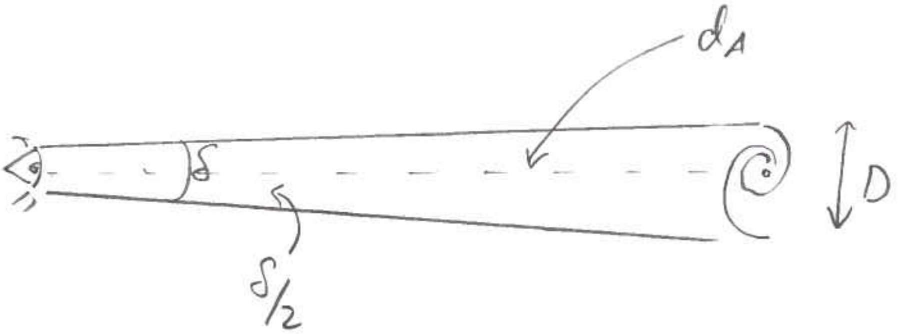
$$l = \frac{L_{received}}{4\pi a_0^2 r_1^2} = \frac{(1+z_1)^{-2} L_{emitted}}{4\pi a_0^2 r_1^2}$$

Now:
$$d_L = \sqrt{\frac{L_{emitted}}{4\pi l}} = \sqrt{a_0^2 r_1^2 (1+z_1)^2}$$

$$= a_0 r_1 (1+z_1).$$

24 Angular diameter distance:

- observed quantity $\longrightarrow$ angular diameter δ.
- intrinsic quantity $\longrightarrow$ diameter D.



Euclidean definition

We can express

$$\tan\frac{\delta}{2} = \frac{D/2}{d_A}.$$

As angle in astrophysics are very small.

$$\tan\frac{\delta}{2} \simeq \frac{\delta}{2}$$

$$\Rightarrow \frac{\delta}{2} = \frac{D}{2d_A} \Rightarrow \delta = \frac{D}{d_A} \Longleftrightarrow d_A = \frac{D}{\delta}.$$

We use this final relation as our definition of angular diameter distance.

Cosmological effects: Centering our coordinates at the observer. When we measure the distance across the diameter of the galaxy we only need the θ component of the metric. This means

$dt = dr = d\phi = 0$ and

$$D = \int_{-\delta/2}^{\delta/2} \sqrt{g_{22}}\, d\theta .$$

$$= \int_{-\delta/2}^{\delta/2} a_1 r_1 \, d\theta .$$

$$= a_1 r_1 \int_{-\delta/2}^{\delta/2} d\theta$$

$$= a_1 r_1 \delta$$

$$\Rightarrow d_A = \frac{D}{\delta} = a_1 r_1 .$$

using $1 + z_1 = \frac{a_0}{a_1}$

we find

$$d_A = (1 + z_1)^{-1} a_0 r_1$$

recall $d_L = (1+z_1) a_0 r_1$

2.5 Redshift relation (comes up in past paper)
The interesting application of these distance is in the context of concrete cosmological models, luminosity distance redshift relation in a matter dominated universe with $k=0$ and $\Lambda=0$. How do we find r_1? If $k=0$.

$$\sigma(r) = \int_0^r \frac{ds}{\sqrt{1-ks^2}}$$

$$= \int_0^r ds$$

$$= r.$$

$$r_1 = \sigma(r_1)$$

$$= \int_{t_1}^{t_0} \frac{dt'}{a(t')}$$

$$= \int_{t_1}^{t_0} \frac{dt'}{a_{const}\, t'^{2/3}}$$

$$= a_{const}^{-1} \Big[3 (t)^{1/3} \Big]_{t_1}^{t_0}$$

$$= a_{const}^{-1} \Big[3t_0^{1/3} - 3t_1^{1/3} \Big]$$

$$= 3a_{const}^{-1} \left(t_0^{1/3} - t_1^{1/3} \right)$$

$$= 3a_{const}^{-1} t_0^{1/3} \left(1 - \frac{t_1^{1/3}}{t_0^{1/3}} \right)$$

Now we use the following observation

$$\left(\frac{t_1}{a_0}\right)^{2/3} = \frac{a_1}{a_0} = (1+z_1)^{-1}$$

Very important!!

$\Downarrow$

Remember in the exam!!

Therefore, we have.

$$r_1 = 3a_{const}^{-1} t_0^{1/3} \left[1 - (1+z_1)^{-1/2} \right]$$

$$= \frac{3 \; t_0}{a_{const} \; t_0^{2/3}} \left[1 - (1+z_1)^{-1/2} \right]$$

$$\Rightarrow d_L = (1+z_1) a_0 r_0 .$$

$$= (1+z) \cancel{a_0} \left[\frac{3t_0}{\cancel{a_{const} \, t_0^{2/3}}} \left[1 - (1+z_1)^{-1/2} \right] \right]$$

Comes from $a_0 = a(t) = a_{const} \, t_0^{2/3}$

Next use $H_0 = \dfrac{\dot{a}(t_0)}{a(t_0)} = \dfrac{2}{3} \dfrac{1}{t_0}$.

$$\Rightarrow d_L = (1+z_1)\frac{2}{H_0}\left[1-(1+z_1)^{-\frac{1}{2}}\right]$$

$$= \frac{2}{H_0}\left[(1+z_1) - \sqrt{1+z_1}\right]$$

Let us consider this relation assuming $z \ll 1$.

$$\sqrt{1+z_1} \simeq 1 + \frac{1}{2}z_1 + \dots$$

$$\Rightarrow d_L = \frac{2}{H_0}\left[1+z_1 - \left(1+\frac{1}{2}z_1\right)\right]$$

$$\Rightarrow d_L = \frac{2}{H_0}\left[z_1 - \frac{1}{2}z_1\right]$$

$$= \frac{z_1}{H_0}$$

$$\Rightarrow \boxed{z_1 = H_0 d_L} \longleftarrow \text{get this correctly.}$$

Which is Hubble's law.

2.6 Contents of the universe

For small redshifts $z \ll 1$ Hubble's law determines the relative velocity of a pair of nearby comoving observers. (Look at 2007 paper)

$$z = H\,dr.$$

If distances of galaxies were accurately known, one could measure H_0 quite precisely. The subscript 0 in H_0 always refers to today's value in cosmology. There are many uncertainties in H_0 these are usually parametrised by h. defined by

$$H_0 = 100h \frac{km}{s Mpc}.$$

Observations suggest $0.5 < h < 0.8$ with $h \simeq 0.7$ being the "standard".

The inverse of the Hubble parameter gives a time scale, the Hubble time

$$H_0^{-1} \simeq 9.78 h^{-1} \text{ Gyr}$$

and the hubble distance

$$cH_0^{-1} \simeq 2998 h^{-1} \text{Mpc}.$$

Most of the mass of the universe does not interact with radiation (neither emits or absorbs), it appears to be dark as opposed to luminous. Various observations suggest

$$0.3 < \Omega < 0.5.$$

The value of the cosmological constant Λ can also be found observationally, for example

by observing distinict type I_a supernovae.
For flat $k=0$ modal, best fit data gives:

$$\Omega_0 \simeq 0.3$$

and $\Omega_\Lambda \simeq 0.7$ so that

$$\Omega_\Lambda + \Omega_0 = 1$$

Gravitional lensing gives:

$$\Omega_\Lambda < 0.74.$$

at 2σ confidence

Also large-scale structure supports this limit and hence we can safely work with

$$0.3 < \Omega_0 < 0.4$$

Ordinary matter in cosmology is referred to as baryons since protons and neutros account for all its density. The baryon is denoted by Ω_b

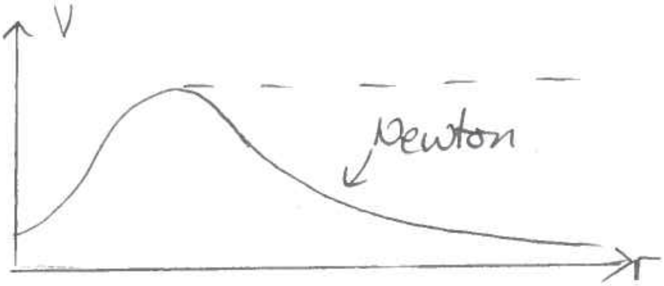
Theoretical calculations of nucleosynthesis yield the following limit.

$$0.010 \leq \Omega_b h^2 \leq 0.022.$$

Clusters of galaxies observations may constrain the ratio Ω_b/Ω_0.

Assume this represents the entire universe we find

$$\frac{\Omega_b}{\Omega_0} = 0.14^{+0.08}_{-0.04}\left(\frac{h}{0.5}\right)^{-3/2}.$$



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Problem Class Sheet 2.

2. $T_{ab} = (\rho + p) u_a u_b + p g_{ab}$

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

$$G_{ab} = 8\pi T_{ab} - \Lambda g_{ab}$$

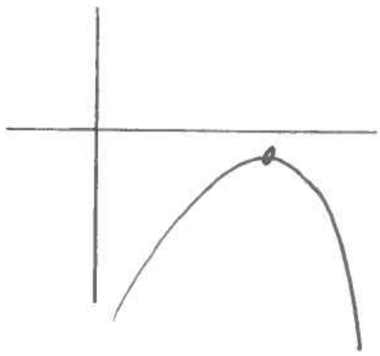
$$= 8\pi \left(T_{ab} - \frac{\Lambda}{8\pi} g_{ab} \right)$$

$$= 8\pi \left(T_{ab} + T^{\Lambda}_{ab} \right)$$

$$T^{\Lambda} = -\frac{\Lambda}{8\pi} g_{ab} \Rightarrow p = -\frac{\Lambda}{8\pi} = -\rho.$$

Sheet 3

1) $k = 0$, $P = \omega\rho$.



$$\rho(t) = \rho_0 + \epsilon\delta\rho_1(t)$$
$$a(t) = a_0 + \epsilon\delta a_1(t).$$

Conservation equation

$$\dot\rho(t) + \frac{3\dot a(t)}{a(t)} \left(\rho(t) + p(t) \right) = 0.$$

$$\dot\rho(t) + \frac{3\dot a(t)}{a(t)} (1+\omega) \rho(t) = 0.$$

$$\dot{\rho}(t) = \epsilon\delta\dot{\rho}_1$$

$$\dot{a}(t) = \epsilon\delta\dot{a}_1$$

$$\Rightarrow \epsilon\delta\dot{\rho}_1 + \frac{\epsilon\delta\dot{a}_1}{(a_0+\epsilon\delta a_1)}(1+\omega)(\rho_0+\epsilon\delta\rho_1) = 0.$$

$$\frac{1}{a+\epsilon\delta a_1} \simeq \frac{1}{a_0\left(1+\epsilon\frac{\delta a_1}{a_0}\right)}$$

$$= \frac{1}{a_0}\left(1-\epsilon\frac{\delta a_1}{a_0}\right)$$

$$\epsilon\delta\dot{\rho}_1 + \epsilon\delta\dot{a}_1(1+\omega)(\rho_0+\epsilon\delta\rho_1)\cdot\frac{1}{a_0}\left(1-\frac{\epsilon\delta a_1}{a_0}\right) = 0.$$

$$\epsilon\delta\dot{\rho}_1 + \epsilon\delta\dot{a}_1(1+\omega)\rho_0\frac{1}{a_0} = 0 + \mathcal{O}(\epsilon^2).$$

$$\delta\dot{\rho}_1 = -\frac{\rho_0}{a}(1+\omega)\delta\dot{a}_1$$

$$-\frac{\ddot{a}}{a} = \frac{4\pi}{3}\rho(1+3\omega) - \frac{\Lambda}{3}$$

$$-\frac{\epsilon\delta\ddot{a}_1}{a_0\left(1+\epsilon\frac{\delta a_1}{a_0}\right)} = \frac{4\pi}{3}(\rho_0+\epsilon\delta\rho_1)(1+3\omega) - \frac{\Lambda}{3}$$

$$+\epsilon\frac{4\pi}{3}\delta\rho_1(1+3\omega)$$

$$-\frac{\varepsilon}{a_0}\delta\ddot{a} = \underbrace{\frac{4\pi}{3}\rho_0\left(1+3\omega\right) - \frac{\Lambda}{3}}_{=0} + \varepsilon\frac{4\pi}{3}\delta\rho_1(1+3\omega)$$

$$\Rightarrow \Lambda = 4\pi\rho_0(1+3\omega).$$

$$\Rightarrow -\delta\ddot{a} = \frac{4\pi}{3}a_0(1+3\omega)\delta\rho_1$$

We have $\delta\rho_1 = -\frac{\rho_0}{a}(1+\omega)\delta a_1$

$$\Rightarrow \delta\ddot{a} = \frac{4\pi}{3}\cancel{a_0}(1+3\omega)\frac{\rho_0}{\cancel{a_0}}(1+\omega)\delta a_1$$

$$\Rightarrow \delta\ddot{a} \propto (1+3\omega)(1+\omega)\delta a_1.$$

Sheet 3, problem 3 (shows up in exam)

$\Lambda = 0$; matter dominated.

$$3H^2 = 8\pi\rho_{total} - \frac{3k}{a^2}$$

$$1 = \underbrace{\frac{8\pi\rho}{3H^2}}_{\Omega} - \frac{k}{a^2H^2}$$

$$\Rightarrow 1 = \Omega - \frac{k}{a^2H^2}$$

$$\Omega = \frac{8\pi\rho}{3H^2}$$

$$\frac{\Omega_1}{\Omega_0} = \frac{\frac{8\pi}{3}\frac{\rho_1}{H_1^2}}{\frac{8\pi}{3}\frac{\rho_0}{H_0^2}} = \frac{\rho_1}{\rho_0}\frac{H_0^2}{H_1^2}$$

$$\frac{k}{a^2H^2} = \Omega - 1 \Rightarrow k = (\Omega - 1)a^2H^2$$

$$\Rightarrow (\Omega_0 - 1)a_0^2 H_0^2 = (\Omega_1 - 1)a_1^2 H_1^2$$

matter dom. $\rho \propto \frac{1}{a^3}$

$$\Leftrightarrow \frac{H_0^2}{H_1^2} = \frac{(\Omega_1 - 1)a_1^2}{(\Omega_0 - 1)a_0^2}$$

$$\frac{\rho_0}{\rho_1} = \frac{a_1^3}{a_0^3}$$

$$\Rightarrow \frac{\Omega_1}{\Omega_0} = \frac{a_0^3}{a_1^3}\cdot\frac{H_0^2}{H_1^2} = \frac{a_0^{\cancel{3}}}{a_1^{\cancel{3}}}\frac{(\Omega_1 - 1)}{(\Omega_0 - 1)}\frac{\cancel{a_1^2}}{\cancel{a_0^2}}$$

$$\frac{\Omega_1}{\Omega_0} = \frac{(\Omega_1 - 1)}{(\Omega_0 - 1)}\frac{a_0}{a_1} = \frac{(\Omega_1 - 1)}{(\Omega_0 - 1)}\frac{1}{z+1}$$

Solve for $\underline{\underline{\Omega_0}}$

$$(1+z)\Omega_1(\Omega_0 - 1) = \Omega_0(\Omega_1 - 1)$$

$$\Omega_0(\Omega_1(1+z) - (\Omega_1 - 1)) = (1+z)\Omega_1$$

$$\Omega_0(\Omega_1 z + 1) = (1+z)\Omega_1$$

$$\Rightarrow \boxed{\Omega_0 = \frac{(1+z)\Omega_1}{1+z\Omega_1}}$$

$$\frac{H_0^2}{H_1^2} = \frac{\Omega_1 - 1}{\Omega_0 - 1}\frac{a_1^2}{a_0^2}$$

$$H_0^2 = (1+z)^2 \frac{\Omega_1 - 1}{\Omega_0 - 1} H_1^2$$

$$\Omega_0 - 1 = \frac{(1+z)\Omega_1 - (1+z\Omega_1)}{1+z\Omega_1}$$

$$= \frac{\Omega_1 - 1}{1+z\Omega_1}$$

$$\Rightarrow H_0^2 = H_1^2 (1+z)^2 \frac{\cancel{(\Omega_1 - 1)}(1+z\Omega)}{\cancel{(\Omega_1 - 1)}}$$

$$= H_1^2 (1+z)^2 (1+z\Omega_1)$$

[Appears in Exams !!!]

Back to lectures:

$$\frac{\Omega_b}{\Omega_0} = 0.14^{+0.08}_{-0.04}\left(\frac{h}{0.5}\right)^{-3/2}.$$

Together with nucleosynthesis bounds one can constrain Ω_0.

The dark matter can either be in the the form of baryonic (for instance cold dark matter) or non-baryonic dark matter. These are many (very many) dark matter models around. Some keywords are massive neutrinos, axion, lightest susy particle, or geometrical generalisations

The present value of the critical density is $\rho_{c,0} = 1.88 h^2 \times 10^{-29}\, g/cm^3$ which is roughly 1 hydrogen atom per cubic meter.

2.7 Temperture.
In summary, you need to remember one equation

2.7.1 Radiation dominated universe.
Liouville's theorem of cassical mechanics can be applied to the phase space of cosmic matter. This allows us to relate the expansion parameter to the temperture.

$$\frac{T(t)}{T(t_1)} = \frac{a(t_1)}{a(t_0)}$$

Since we know the relation between $a(t)$ and the redshifts we have:

$$\frac{T(t)}{T(t_1)} = 1+z.$$

2.7.2 Matter dominated universe

In a matter epoch we have:

$$\frac{T(t)}{T(t_1)} = \frac{a(t_1)^2}{a(t)^2}$$
$$= (1+z)^2$$

At a temperature of about $0.3 \sim 3.6 \times 10^3\,K$ photons decouple from matter and hydrogen is formed, so-called decoupling or recombination. Hence, the redshift is:

$$z_{rec} = \frac{T(t_{rec})}{T(t_1)} - 1$$

If we interpret the cosmic microwave background radiation as the cooled gas of photons, then we can take

$$T(t_1) \sim 2.73\,K.$$

$$\Rightarrow z_{rec} \simeq 1330.$$

Lets us compare this with the temperature of matter

$$\frac{T(t)}{T(t_1)} = (1+z)^2 .$$

$$T(t_1) = \frac{T(t)}{(1+z)^2} = \frac{3.6 \times 10^3 K}{(1+1330)^2}.$$

$$= 2 \times 10^{-3} K .$$

Hence, the matter components cools down much faster than the radiation.

History of the Universe.

z	t/s	Description	T/k	E/eV
10^{32}	10^{-43}	Quantum Cosmology		
	10^{-42}	Inflation begins?		10^{25}
10^{26}	10^{-35}	GUT phase transition $SU(3) \times SU(2) \times U(1)$ breaks down to $SU(2) \times U(1)$	10^{26}	
	10^{-32}	Inflation ends		
	$2\times10^{-12}/10^{-11}$	Electroweak phase transitions $SU(2) \times U(1)$ breaks to $U(1)_{em}$	10^{15}	100G.
	$10^{-1}/10^{0}$	neutrons decouple e^+e^- anihilation γ and a few necleons were in thermal equlibrum		

z	t/s	Description	T/k	E/eV
	$10^3/10^4$	Radiation dominated		
$10^5/5\times10^3$	10^4 Y	Matter radiation equilibrium	10^4	1
1.2×10^3	10^5 Y	photon decoupling.	3.6×10^3	0.1.
10-30	10^9 Y	galaxy formation		
5	3×10^9 Y	star formation		
	1.5×10^{10} Y	planet formation		
		$T_\gamma = 2.75K$		
		$T_\nu = 1.96K$		10^{-4}

3 Inflation

3.1 Shortcomings of standard cosmology.

3.3.1 Flatness problem

In terms of the density parameter Ω, the time-time component of the field equation is

$$\Omega - 1 = \frac{k}{a^2H^2}.$$

If the constant time hypersurfaces are flat ($k=0$), then $\Omega = 1$ and it remains so for all time.

If $k \neq 0$, the density parameter evolves. In a nearly flat universe, we derived (matter dominated).

$$a \sim t^{2/3} \quad H \sim \frac{1}{t}$$

$$\Rightarrow aH \sim \frac{t^{2/3}}{t} \sim t^{-1/3}$$

(radiation dominated)

$$a \sim t^{1/2} \quad H \sim \frac{1}{t} \Rightarrow aH \sim \frac{t^{1/2}}{t} \sim t^{-1/2}$$

$$\Omega - 1 = \frac{k}{a^2H^2}$$

Therefore the density parameter evolves as follows:

$$|\Omega - 1| = \begin{cases} t^{2/3} & \text{matter dom} \\ t & \text{radiation dom.} \end{cases}$$

The flatness problem is that $1/aH$ is in general an increasing function. We observe Ω_0 of the order of unity and therefore Ω was extermely close to 1 at early times. For instance at nucleosynthesis we must require

$$|\Omega(t_{\text{nucleosynthesis}}) - 1| < 10^{-16}.$$

Such fine-tuned initial conditions are problematic as they are unlikely.

3.12 Horizon problem.

We discussed the possible presence of particle horizon depending on the convergence of

$$\eta = \int \frac{dt}{a(t)}$$

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$$\eta = \int \frac{dt'}{a(t')}$$

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\underline{x}^2\right]$$

In a radiation dominated and also in a matter dominated epoch there exist particle horizon.

On the other hand, the cosmic microwave background radiation is nearly homogenous. However, the similarly looking region cannot have interacted before recombination. This suggests we need finely tuned initial conditions to get the observations.

Homogeneneity and isotrogy problem
We discussed that the large scale homogeneity and isotropy might be encoded in the initial conditions. The other key questions is ~~can~~ we explain the inhomogeneities?

In particular the inhomogencities should have a much longer scale than the corresponding horizon scale. We would need fine tuned initial conditions to explain this

3.2 Accelerated expansion - inflation.
A cosmological epoch where the universe is accelerating is called inflation. Formally, we can define by

$$\boxed{\ddot{a} > 0.}$$

This can be written more physically in the following way:

$$H = \frac{\dot{a}}{a} \Leftrightarrow \dot{a} = Ha.$$

$$\frac{d}{dt}\left(\frac{1}{\dot{a}}\right) = \frac{d}{dt}\left(\frac{H^{-1}}{a}\right)$$

$$= -\frac{1}{\dot{a}^2}\ddot{a}$$

$$\Rightarrow \ddot{a} > 0 \Leftrightarrow \frac{d}{dt}\left(\frac{H^{-1}}{a}\right) < 0.$$

which we can also use as a definition for inflation.

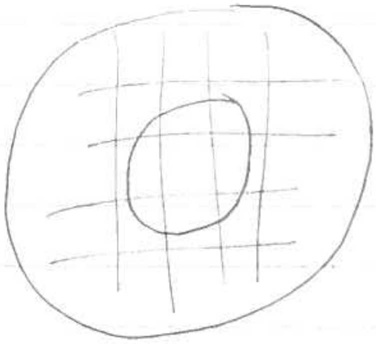
The quantity H^{-1}/a is the comoving Hubble length. This is an imporant scale as it determines if two regions can communicate now.

If they are seperated by distances greater than H^{-1}/a they cannot communicate

Note that the particle horizon (sometimes called comiving horizon) seperates two regions that never could have communicated.

In principle this allows the following: The length scale of the particle horizon could be much large then the comoving Hubble length today, particles cannot communicate today but were in contact at early times. Actually, during inflation the observable universe becomes smaller. The characterisic scale occupies a smaller coordinate size.

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\underline{x}^2\right]$$
$$= -dt^2 + a^2(t)\, d\underline{x}^2 .$$



A particular combination of the field equations was ($\Lambda = 0$)

$$-\frac{\ddot{a}}{a} = \frac{4\pi}{3}(\rho + 3p)$$

$$\Rightarrow \ddot{a} > 0 \Leftrightarrow \rho + 3p < 0$$

Since $\rho > 0$ acceleration requires negative pressure.

If we assume $P = w\rho$ the

$$\rho + 3p = \rho(1+3w)$$

$$1+3w < 0 \Rightarrow w < -1/3$$

3.3 Solving the problem:

Flatness:

$$\frac{d}{dt}\left(\frac{1}{Ha}\right) < 0.$$

Hence $|\Omega - 1|$ is driven towards 0 horizon.
The horizon problem is sloved because we can reduce the comoving Hubble length.

3.4 Scalar field in cosmology.
Spin 0 particles or scalar particles are describle by the following Lagrangian.

$$\mathcal{L} = -\frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi).$$

which leads to the energy-momentum tensor

$$T_{ab} = \nabla_a \phi \nabla_b \phi + g_{ab} \mathcal{L}$$

—/—

$$\begin{aligned} \nabla_a \nabla^a \phi &= \nabla_a (\partial^a \phi) \\ &= \partial_a \partial^a \phi + \Gamma^a_{ab} \partial^b \phi \\ &\neq \partial_a \partial^a \phi. \end{aligned}$$

In cosmology we assume $\phi = \phi(t)$. We will generally assume $k=0$.

$$ds^2 = -dt^2 + a^2(t)\left[dx^2 + dy^2 + dz^2\right].$$

Our Lagrangian is

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

[recall harmonic oscillator].

$$T_{ab} = \nabla_a \phi \nabla_b \phi + g_{ab}\mathcal{L}$$

$$T_{tt} = \partial_t \phi \partial_t \phi + (-1)\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right]$$

$$= \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$T_{xx} = T_{yy} = T_{zz} = 0 + a^2(t)\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right]$$

$$= a^2(t)\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right]$$

We compare with a prefect fluid

$$T_{ab} = \mathrm{diag}(\rho, a^2 P, a^2 P, a^2 P)$$

and this implies that we can write:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi).$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

The scalar field during inflation is often called inflaton. Different models are characterised by different potentials.

Using $P_\phi = w_\phi \rho_\phi$

$$\Rightarrow w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}$$

The field equations are

$$H^2 = \frac{8\pi}{3}\rho.$$

$$-\frac{\ddot{a}}{a} = \frac{4\pi}{3}(\rho + 3P).$$

and

$$\dot{\rho} + 3H(\rho + P) = 0.$$

If $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$

$$\Rightarrow \dot{\rho} = \frac{1}{2}\cdot 2\dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi}.$$

$$\Rightarrow \dot{\rho} + 3H\left(\rho + P\right) = 0.$$

$$\Leftrightarrow \dot{\phi}\ddot{\phi} + \frac{dV}{d\phi}\dot{\phi} + 3H\left(\frac{1}{2}\dot{\phi}^2 + V + \frac{1}{2}\dot{\phi}^2 - V\right)$$

$$= \dot{\phi}\left[\ddot{\phi} + \frac{dV}{d\phi} + 3H\dot{\phi}\right] = 0.$$

If $\dot{\phi} \neq 0$

$$\underbrace{\ddot{\phi} + 3H\dot{\phi}}_{\nabla^a \nabla_a \phi} + \frac{dV}{d\phi} = 0.$$

This is called the scalar wave equation. It is common to rename the coupling constant in the field equation.

$$M_{pl} = \frac{1}{\sqrt{8\pi}}$$

which is the reduced Planck mass.

Using this the field equation:

$$H^2 = \frac{1}{3}\frac{1}{M_{pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]$$

$$-\frac{\ddot{a}}{a} = \frac{1}{3}\frac{1}{M_{pl}^2}\left[\dot{\phi}^2 - V(\phi)\right]$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

One see that:

$$V(\phi) > \dot{\phi}^2$$

gives inflation.

If the potential is sufficiently flat, then this condition is satisfied. Even if it is not obeyed intially this quickly changes if the field is away from the potential minimum.

3.5 Slow-roll inflation.

The slow-roll approximation in inflation:

$$\ddot{\phi} \ll 1 \qquad \frac{1}{2}\dot{\phi}^2 \ll V(\phi)$$

In this approximation the field equation:

$$H^2 \simeq \frac{V(\phi)}{3M_{pl}^2}$$

$$3H\dot{\phi} \simeq -\frac{dV}{d\phi}$$

Two convenient parameter are the so-called slow roll parameters.

$$\epsilon(\phi) = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2$$

$$\eta(\phi) = M_{pl}^2\left(\frac{V''}{V}\right)$$

The slow-roll approximation is valid if $\epsilon \ll 1$ and $\eta \ll 1$.

We have

$$H = \frac{\dot{a}}{a} \quad \text{and} \quad \dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2}$$

$$= \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}$$

$$= \frac{\ddot{a}}{a} - H^2 .$$

Inflatcon means $\ddot{a} > 0$.

$$\Leftrightarrow \dot{H} + H^2 > 0$$
$$\Leftrightarrow \dot{H} > -H^2$$
$$\Leftrightarrow -\dot{H}/H^2 < 1$$

$$H^2 \simeq \frac{V(\phi)}{3M_{pl}^2}.$$

$$2H\dot{H} \simeq \frac{1}{3M_{pl}^2} \frac{dV}{d\phi} \dot{\phi}.$$

$$\Rightarrow \dot{H} \simeq \frac{1}{6HM_{pl}^2} \dot{\phi} \frac{dV}{d\phi} \simeq \frac{1}{6HM_{pl}^2} \frac{dV}{d\phi} \left(-\frac{dV}{d\phi} \frac{1}{3H} \right)$$

$$= \frac{1}{18} \frac{1}{M_{pl}^2} \cdot \frac{1}{H^2} (-) (V')^2$$

$$\Rightarrow -\frac{\dot{H}}{H^2} = \frac{1}{18} \cdot \frac{1}{M_{pl}^2} \cdot \frac{1}{H^4} (V')^2$$

$$\Rightarrow -\frac{\dot{H}}{H^2} \simeq \frac{1}{18} \cdot \frac{1}{M_{pl}^2} \frac{9 M_{pl}^4}{V^2} V'^2 .$$

$$= \frac{1}{2} M_{pl}^2 \left(\frac{V'}{V}\right)^2 = \varepsilon .$$

<u>Example</u> $V = \frac{1}{2} m^2 \phi^2$

$$\varepsilon = \frac{1}{2} M_{pl}^2 \left(\frac{\cancel{\frac{1}{2}} \cancel{m^2} 2\phi}{\cancel{\frac{1}{2}} \cancel{m^2} \phi^2} \right)^2 = \frac{1}{2} M_{pl}^2 \frac{4}{\phi^2}$$

$$= 2 \left(\frac{M_{pl}}{\phi} \right)^2 \ll 1 \Longleftrightarrow \text{inflation} .$$

$$\eta = M_{pl}^2 \frac{V''}{V} = M_{pl}^2 \frac{m^2}{\frac{1}{2} m^2 \phi^2} = 2 \left(\frac{M_{pl}}{\phi} \right)^2$$

<u>3.6 e-folding</u>

The ratio of the scale factor at the end of inflation to its value at some time t is used to measure the amount of inflation occured. Typically this about 10^{26}. This suggest we sould define

$$N(t) = \log \frac{a(t_{end})}{a(t)}$$

The number of e-folding is taken to be 50-60. $N(t)$ measure the number of e-foldings that still have to happen.

We can calculate N without using the field equation if we work in the slow-roll approximation.

$$N = \log \frac{a(t_{end})}{a(t)}$$

$$H = \frac{\dot{a}}{a} = \frac{d}{dt}(\log a)$$

$$= \int_t^{t_{end}} H(t')\, dt'$$

$$\simeq \int_t^{t_{end}} \sqrt{\frac{V}{3M_{pl}^2}}\, dt$$

$$= \int_\phi^{\phi_{end}} \sqrt{\frac{V}{3M_{pl}^2}} \frac{dt}{d\phi}\, d\phi.$$

$$= \int_\phi^{\phi_{end}} \sqrt{\frac{V}{3M_{pl}^2}} \frac{1}{\dot{\phi}}\, d\phi.$$

$$\simeq \int_\phi^{\phi_{end}} \sqrt{\frac{V}{3M_{pl}^2}} \left(-\frac{3H}{V'}\right) d\phi$$

$$\simeq \int_\phi^{\phi_{end}} \frac{V}{\cancel{3}M_{pl}^2} \left(-\frac{\cancel{3}}{V'}\right) d\phi$$

$$\Rightarrow N \simeq \frac{1}{M_{pl}^2} \int_{\phi_{end}}^{\phi} \frac{V}{V'}\, d\phi$$

Example $V = \frac{1}{2} m^2 \phi^2 \quad \varepsilon = 2\left(\frac{M_{pl}}{\phi}\right)^2$

Define the end of inflation by $\varepsilon = 1$

$$\Rightarrow 2\left(\frac{M_{pl}}{\phi_{end}}\right)^2 = 1$$

$$\Leftrightarrow \phi_{end} = \sqrt{2}\, M_{pl}.$$

$$N(t_i) = \frac{1}{M_{pl}^2}\int_{\phi_{end}}^{\phi_i} \frac{\frac{1}{2}m\phi^2}{\frac{1}{2}m2\phi}\, d\phi = \frac{1}{2M_{pl}^2}\left[\frac{1}{2}\phi^2\right]_{\phi_{end}}^{\phi_i}.$$

$$= \frac{1}{4}\frac{1}{M_{pl}^2}\left[\phi_i^2 - 2M_{pl}^2\right]$$

$$= \left(\frac{\phi_i}{2M_{pl}}\right)^2 - \frac{1}{2}.$$

Choosing $\phi_i = 16 M_{pl}$ gives $N \sim 63$.

3.7 Power-law inflation

Power law inflation is a solution to the equations with potential:

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{p}}\,\frac{\phi}{M_{pl}}\right)$$ [Fav. Exam question]

↓

Was in the past papers.

Let's slove the slow roll field equations

① $H^2 \simeq \frac{V}{3M_{pl}^2}$

$$\simeq \frac{V_0}{3M_{pl}^2} \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right)$$

$$\Rightarrow H \simeq \frac{\sqrt{V_0}}{\sqrt{3}M_{pl}} \exp\left(-\frac{1}{2}\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right)$$

② $3H\dot{\phi} \simeq -V'$

$$3\dot{\phi}\frac{\sqrt{V_0}}{\sqrt{3}M_{pl}} \exp\left(-\frac{1}{2}\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right)$$

$$\simeq -V_0\left(-\sqrt{\frac{2}{p_0}}\frac{1}{M_{pl}}\right) \exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right)$$

4/3/1 +

$$V(\phi) = V_0 \exp\left(-\sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right)$$

Field eqns

$$\begin{cases} H^2 \simeq \dfrac{V_0}{3M_{pl}^2} \exp\left(-\sqrt{\dfrac{2}{p}} \dfrac{\phi}{M_{pl}}\right) \\ 3H\dot{\phi} \simeq + V_0 \sqrt{\dfrac{2}{p}} \dfrac{1}{M_{pl}} \exp\left(-\sqrt{\dfrac{2}{p}} \dfrac{\phi}{M_{pl}}\right) \end{cases}$$

$$H \simeq \frac{\sqrt{V_0}}{\sqrt{3} M_{pl}} \cdot \exp\left(-\frac{1}{2} \sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right)$$

Now we can eliminate H from the second equation

$$3 \frac{\sqrt{V_0}}{\sqrt{3} \cancel{M_{pl}}} \exp\left(-\frac{1}{2} \sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right) \dot{\phi} \simeq V_0 \sqrt{\frac{2}{p}} \frac{1}{\cancel{M_{pl}}} \cdot$$

$$\cdot \exp\left(-\sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right)$$

$$\frac{\sqrt{3}\sqrt{V_0}}{V_0} \sqrt{\frac{p}{2}} \dot{\phi} \simeq \exp\left(-\sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right) \exp\left(\frac{1}{2} \sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right)$$

$$\frac{\sqrt{3}}{\sqrt{V_0}} \sqrt{\frac{p}{2}} \dot{\phi} \simeq \exp\left(-\frac{1}{2} \sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right)$$

$$\frac{\sqrt{3}}{\sqrt{V_0}} \sqrt{\frac{p}{2}} \frac{d\phi}{dt} \simeq \exp\left(-\frac{1}{2} \sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right)$$

$$\exp\left(\frac{1}{2}\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right) d\phi \simeq \frac{\sqrt{V_0}}{\sqrt{3}} \cdot \sqrt{\frac{2}{p}}\, dt.$$

$$2M_{pl}\sqrt{\frac{p}{2}}\exp\left(\frac{1}{2}\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right) \simeq \frac{\sqrt{V_0}}{\sqrt{3}}\sqrt{\frac{3}{p}}(t+t_i)$$

$$\exp\left(\frac{1}{2}\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right) \simeq \sqrt{\frac{V_0}{3}}\frac{2}{p}\frac{1}{2M_{pl}}(t+t_i)$$

$$\frac{1}{2}\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}} \simeq \log\left(\sqrt{\frac{V_0}{3}}\frac{2}{p}\frac{1}{2M_{pl}}(t+t_i)\right)$$

$$\Leftrightarrow \frac{\phi}{M_{pl}} \simeq \sqrt{2p}\log\left(\sqrt{\frac{V_0}{3}}\frac{1}{p}\frac{1}{M_{pl}}(t+t_i)\right)$$

We are now using this result to find $a(t)$.

$$H^2 \simeq \frac{V}{3M_{pl}^2} = \frac{V_0}{3M_{pl}^2}\exp\left(-\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right)$$

We found

$$\exp\left(\frac{1}{2}\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right) \simeq \sqrt{\frac{V_0}{3}}\frac{1}{p}\frac{1}{M_{pl}}(t+t_i)$$

$$\exp\left(\sqrt{\frac{2}{p}}\frac{\phi}{M_{pl}}\right) \simeq \frac{V_0}{3} \cdot \frac{1}{p^2} \cdot \frac{1}{M_{pl}^2}(t+t_i)^2.$$

$$\Rightarrow H^2 \simeq \frac{\cancel{V_0}}{\cancel{3M_{pl}^2}} \frac{\cancel{3}p^2\cancel{M_{pl}^2}}{\cancel{V_0}}(t+t_i)^{-2}$$

Comes in Exams!!!

$$\Rightarrow H \simeq \frac{p}{(t+t_i)}$$

$$\frac{\dot{a}}{a} \simeq \frac{p}{(t+t_i)}$$

$$(\log a)^{\cdot} \simeq \frac{p}{t+t_i}$$

$$(\log a)^{\cdot} \simeq \frac{p}{t+t_i}$$

$$\log a \simeq p \log (t+t_i) + C$$

$$a(t) \simeq a_0 (t+t_i)^p$$

Let us compute the slow roll parameter for this pontential

$$V = V_0 \exp\left(-\sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right)$$

$$V' = V_0 \left(-\sqrt{\frac{2}{p}} \frac{1}{M_{pl}}\right) \exp\left(-\sqrt{\frac{2}{p}} \frac{\phi}{M_{pl}}\right)$$

$$= -\sqrt{\frac{2}{p}} \frac{1}{M_{pl}} V.$$

$$V'' = -\sqrt{\frac{2}{p}} \frac{1}{M_{pl}} V' = \left(-\sqrt{\frac{2}{p}} \frac{1}{M_{pl}}\right)^2 V.$$

$$\Rightarrow \varepsilon = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2$$

$$= \frac{M_{pl}^2}{2}\left(-\sqrt{\frac{2}{P}}\ \frac{1}{M_{pl}}\right)^2$$

$$= \frac{M_{pl}^2}{2}\ \frac{2}{P}\ \frac{1}{M_{pl}^2} = \frac{1}{P}$$

$$\eta = M_{pl}^2 \frac{V''}{V} = M_{pl}^2\left(\sqrt{\frac{2}{P}}\ \frac{1}{M_{pl}}\right)^2$$

$$= \frac{2}{P}.$$

If $p > 2$ (or $p > 1$ if we consider $\varepsilon < 1$ only) this model satisfies the conditcon of inflatcon. Note, ε and η are both constants and thus in such a model inflatcon cannot end. This makes this a somewhat unlikely model.

—/—

Problem Class

Sheet 4, P3

Given $V(\phi) = \dfrac{\alpha}{\phi^2}$ $\quad V' = -2\dfrac{\alpha}{\phi^3}$

$$V'' = 6\frac{\alpha}{\phi^4}$$

Slow roll eqns

$$H^2 \simeq \frac{1}{3M_{pl}^2} V(\phi)$$

$$3H\dot{\phi} \simeq -V'(\phi)$$

$$\varepsilon = \frac{1}{2} M_{pl}^2 \left(\frac{V'}{V}\right)^2$$

$$= \frac{1}{2} M_{pl}^2 \left[\frac{-2\frac{\mathcal{K}}{\phi^3}}{\frac{\mathcal{K}}{\phi^2}}\right]^2$$

$$= \frac{1}{2} M_{pl}^2 \left(\frac{4}{\phi^2}\right)$$

$$= \frac{2M_{pl}^2}{\phi^2}$$

$$\eta = M_{pl}^2 \left(\frac{V''}{V}\right) = M_{pl}^2 \left[\frac{\frac{6\mathcal{K}}{\phi^4}}{\frac{\mathcal{K}}{\phi^2}}\right] = 6\frac{M_{pl}^2}{\phi^2}$$

$$\varepsilon = 1 \iff 2M_{pl}^2 = 1$$

$$\boxed{\phi_{end} = \sqrt{2} M_{pl}}$$

$\eta = 1 \iff \phi_{end} = \sqrt{6}\, M_{pl}$

$$N = \frac{1}{M_{pl}^2}\int_{\phi_{end}}^{\phi_i} \frac{V}{V'}\, d\phi = \frac{1}{M_{pl}^2}\int_{\phi_{end}}^{\phi_i} \frac{\left(\frac{\mathcal{K}}{\phi^2}\right)}{\left(-\frac{2\mathcal{K}}{\phi^3}\right)}\, d\phi.$$

$$= \frac{1}{M_{pl}^2}\int_{\phi_{end}}^{\phi_i} -\frac{\phi}{2}\, d\phi$$

$$= \frac{1}{M_{pl}^2}\left[-\frac{1}{4}\phi^2\right]_{\phi_{end} = \sqrt{2} M_{pl}}^{\phi_i}$$

$$= \frac{1}{M_{pl}^2}\left[-\frac{1}{4}\phi_i^2 + \frac{1}{4}\left(\sqrt{2}\, M_{pl}\right)^2\right]$$

$$= \frac{1}{2} - \frac{1}{4}\left(\frac{\phi_i}{M_{pl}}\right)^2$$

N cannot be bigger than $\frac{1}{2}$ and therefore this model does not inflate enough $\Rightarrow$ bad model.

$$H^2 \simeq \frac{1}{3M_{pl}^2}\frac{\mathcal{K}}{\phi^2} \Rightarrow H \simeq \frac{\sqrt{\mathcal{K}}}{\sqrt{2}}\frac{1}{M_{pl}\phi}$$

$$3H\dot{\phi} \simeq +2\frac{\mathcal{K}}{\phi^3}$$

$$\Rightarrow 3 \frac{\sqrt{\mathcal{K}}}{\sqrt{3}} \frac{1}{M_{pl}} \frac{1}{\phi} \dot{\phi} \simeq 2 \frac{\mathcal{K}}{\phi^3}.$$

$$\sqrt{3\mathcal{K}} \frac{1}{M_{pl}} \frac{d\phi}{dt} \simeq 2 \frac{\mathcal{K}}{\phi^2}$$

$$\phi^2 \, d\phi \simeq \frac{2M_{pl}\mathcal{K}}{\sqrt{3\mathcal{K}}} \, dt$$

$$\frac{1}{3}\phi^3 \simeq \frac{2M_{pl}\sqrt{\mathcal{K}}}{\sqrt{3}} \, t + C.$$

$$\phi^3 \simeq 2M_{pl}\sqrt{3\mathcal{K}} \, t + \tilde{C}$$

$$\phi(t) \simeq (\tilde{C} + 2M_{pl}\sqrt{3\mathcal{K}} \, t)^{1/3}$$

optional extra.

$$H \simeq \frac{\sqrt{\mathcal{K}}}{\sqrt{3}} \frac{1}{M_{pl}} \frac{1}{\phi} \simeq \frac{\sqrt{\mathcal{K}}}{\sqrt{3}} \frac{1}{M_{pl}} (\tilde{C} + 2M_{pl}\sqrt{3\mathcal{K}} \, t)^{-1/3}$$

$$(\log a)^{\cdot} \simeq \sqrt{\frac{\mathcal{K}}{3}} \frac{1}{M_{pl}} \frac{1}{(\tilde{C} + 2M_{pl}\sqrt{3\mathcal{K}} \, t)^{1/3}}$$

$$\Rightarrow \log a \simeq \sqrt{\frac{\mathcal{K}}{3}} \frac{1}{M_{pl}} \frac{3}{2} \frac{1}{2M_{pl}\sqrt{3\mathcal{K}}} (\tilde{C} + 2M_{pl}\sqrt{3K} \, t)^{2/3}$$

$$\simeq \frac{1}{4M_{pl}^2} (\tilde{C} + 2M_{pl}\sqrt{3\mathcal{K}} \, t)^{2/3}$$

—/—

4. Introduction to cosmological perturbation theory.

Cosmological perturbation theory is an imporant theory in modern cosmology. It allows us to calculate the following (among other applications)
- the growth of structures in the universe (structure formation)
- the fluctuation of the cosmic microwave background radiation).

We need to understand the evolution of perturbation in an expanding universe.

We will need to introduce the concept of the Lie derivative in order to formulate cosmological perturbation theory.

4.1 Local isomteries and the Lie derivative

Let $p \in M$ a point on the maniflod M. Consider coordinate X^a and a new coordinate system $X^{a'} = X^{a'}(X^b)$. If g_{ab} denotes the meteric on M, then then there are two possible ways to intepret

$$g'_{ab}(X^{c'})$$

In the active picture $p(X^{a'})$ corresponds to another point $p' \in M$ while in the passive picture $X^{a'}$ are the new coordinates of the same point p.

Def 4.1 : Local isometery The transformation $x^a \mapsto x'^a = x'^a = x'^a(x^b)$ is called a symmetry transformation or local isometery if $g'_{ab} = g_{ab}$.

Def 4.2 : Lie derivative of a function. Let $f \in C^{\infty}(M)$ and let ξ^a be a vector, then the Lie derivative of f is defined by

$$L_\xi f = \xi^a \nabla_a f.$$

Note : What is $(L_\xi V)^a$?
Let V^a and W_a be two vectors. Then $V^a W_a$ is a scalar

Thus
$$L_\xi (V^a W_a) = \xi^b \nabla_b (V^a W_a)$$
$$= (\xi^b \nabla_b V^a) W_a + (\xi^b \nabla_b W_a) V^a$$

Next choose $W_a = V_a$.

$$= V_a (\xi^b \nabla_b V^a) + (\xi^b \nabla_b V_a) V^a$$

$$= V_a (\xi^b \partial_b V^a + \xi^b \Gamma^a_{bc} V^c)$$
$$+ V^a (\xi^b \partial_b V_a - \xi^b \Gamma^c_{ba} V_c)$$

$$= V_a \partial_b V^a + \xi^b \Gamma^a_{bc} V^c V_a + V^a \xi^b \partial_b V_a - \xi^b \Gamma^c_{ba} V_c V^a.$$

we stop here.

<u>Def 4.3</u>. Lie derivative of a vector. Let V^a and ξ^a be contravariant vectors. Then the Lie derivative of V^a with respect to ξ^a is defined to be

$$(L_\xi V)^a = \xi^b \partial_b V^a - V^b \partial_b \xi^a .$$

$$\begin{aligned}
L_\xi (V^a V_a) &= V_a (L_\xi V)^a + V^a (L_\xi V)_a \\
&= V_a(\xi^b \partial_b V^a - V^b \partial_b \xi^a) \\
&\qquad + V^a (L_\xi V)_a \\
&= \xi^b \partial_b (V^a V_a)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow V^a (L_\xi V)_a &= \xi^b \partial_b (V^a V_a) \\
&\qquad - V_a(\xi^b \partial_b V^a - V^b \partial_b \xi^a) \\
&= V^a \xi^b \partial_b V_a + \cancel{V_a \xi^b \partial_b V^a} - \cancel{V_a \xi^b \partial_b V^a} \\
&\qquad + V_a V^b \partial_b \xi^a . \\
&= V^a (\xi^b \partial_b V_a + V_c \partial_a \xi^c)
\end{aligned}$$

$$\Rightarrow (L_\xi V)_a = \xi^b \partial_b V_a + V_b \partial_a \xi^b .$$

This definition can be extended to higher rank tensors. For the meteric g_{ab} we have

$$(\mathcal{L}_\xi g)_{ab} = \xi^c \partial_c g_{ab} + g_{cb}\, \partial_a \xi^c + g_{ac}\, \partial_b \xi^c$$

Theorem 4.1: The vector ξ^a generates a local isometry if $(\mathcal{L}_\xi g)_{ab} = 0$.

<u>Killing vectors</u>

Killing vectors are important in cosmology and general relativity because we can use them to define conserved quantities. They are vector fields for which the Lie derivative of the meteric is zero. We know that a necessarly condition for the existence of a local isometery is

$$\xi^c \partial_c g_{ab} + \partial_b \xi^d g_{ad} + \partial_a \xi^c g_{cb} = 0.$$

Let's use $\nabla_c g_{ab} = 0$ to write $\partial_c g_{ab}$

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ca} g_{db} - \Gamma^d_{cb} g_{ad} = 0.$$

$$\Leftrightarrow \partial_c g_{ab} = \underbrace{\nabla_c g_{ab}}_{=0.} + \Gamma^d_{ca} g_{db} + \Gamma^d_{cb} g_{ad}$$

Sub into the Lie derivative

$$\Rightarrow \xi^c (\Gamma^d_{ca} g_{db} + \Gamma^d_{cd})$$

$$+ \partial_b \xi^c g_{ac} + \partial_a \xi^c g_{cb} = 0.$$

$$\xi^c \Gamma^d_{ca} g_{db} + \partial_a \xi^d g_{db} + \xi^c \Gamma^d_{cb} g_{ad} + \partial_b \xi^d g_{ad}$$

$$= g_{db}(\partial_a \xi^d + \Gamma^d_{ca} \xi^c) + g_{ad}(\partial_b \xi^d + \Gamma^d_{cb} \xi^c) = 0.$$

$$= g_{db}(\nabla_a \xi^d) + g_{ad}\nabla_b \xi^d = 0.$$

$$\Leftrightarrow \boxed{\nabla_a \xi_b + \nabla_b \xi_a = 0.}$$

These are the so-called Killing equations.

Since the second covariant derivative involve the Riemann curvature tensor the vector have to satisfy certain integrability conditions. (not discussed).

Example: Consider Minkowski space with metric. $\eta = \mathrm{diag}(-1, +1, +1, +1)$

The Killing equation becomes

$$\partial_a \xi_b + \partial_b \xi_a = 0.$$

which is solved by

$$\xi^c = w^c{}_b X^b + a^c$$

where $w^c{}_b$ is skew-symmetric, $w^c{}_b$ and a^c are constants. Hence, for Minkowki space there are 10 Killing vectors, this corresponds to the Poincare group.

Example: The vector $\xi^a = \delta^a{}_t$ is a Killing vector for the Schwarzchild metric.

— / —

Remarks on conservation:

$\nabla_a T^{ab} = 0$ $\qquad$ ξ_a satisfies Killing

$$\nabla_b(\xi_a T^{ab}) = \nabla_a \xi_a T^{ab} + \underbrace{\xi_a \nabla_b T^{ab}}_{=0}.$$

$$\nabla_b \xi_a + \nabla_a \xi_b = 0$$

$$\Leftrightarrow \nabla_b \xi_a = -\nabla_a \xi_b$$

$$\Rightarrow \nabla_b \xi_a T^{ab} = 0.$$

$$\Rightarrow \nabla_b(\xi_a T^{ab}) = 0.$$

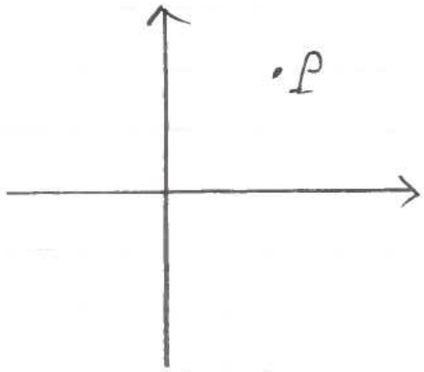
$$\Rightarrow \xi_a T^{ab} = \text{conserved energy}.$$

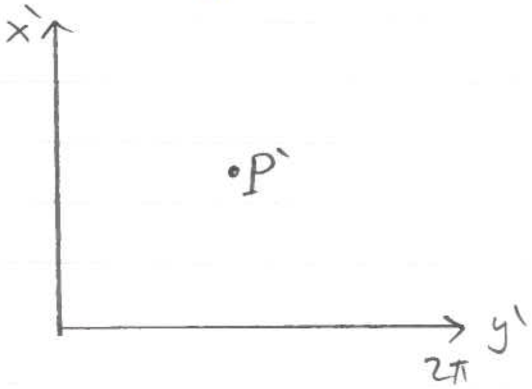
—/—

<u>21/3/14</u>

$(x, y) \in M$
$-\infty \leq x \leq \infty$
$-\infty \leq y \leq \infty$

$(x', y') \in M'$
$0 \leq x' \leq \infty$
$0 \leq y' < 2\pi$





<u>Theorem 4.1</u>: The vector ξ^a generates a local geometery if $(L_\xi g)_{ab} = 0$.

<u>Proof</u>: Let ξ^a generate an infinitesimal transformation

$$X''^a \rightarrow X'^a + \xi^a$$

First, we consider $g'_{ab}(X''^c)$ and compute the Taylor series with respect to ξ^a. This gives

$$f(x+\epsilon) = f(x) + \frac{\partial f}{\partial x}\epsilon + \dots$$

$$g'_{ab}(X''^c) = g'_{ab}(X'^c + \xi^c)$$

$$= g'_{ab}(X'^c) + \xi^c \partial_c g'_{ab}(X'^c) + O(\xi^2)$$

Secondly, we use the tensor transformation properties of g_{ab}, rank 2 covariant tensor.

$$g'_{ab}(x'^{i}) = \frac{\partial x^{c}}{\partial x'^{a}} \frac{\partial x^{d}}{\partial x'^{b}} g_{cd}(x^{i})$$

If $x'^{a} = x^{a} + \xi^{a}$

$$\frac{\partial x^{c}}{\partial x'^{a}} = \frac{\partial(x'^{c} - \xi^{c})}{\partial x'^{a}} = \delta_{a}^{c} - \partial'_{a}\xi^{c}$$

$$\Rightarrow \frac{\partial x^{c}}{\partial x'^{a}} \frac{\partial x^{d}}{\partial x'^{b}} = (\delta_{a}^{c} - \partial'_{a}\xi^{c})(\delta_{b}^{d} - \partial'_{b}\xi^{d})$$

$$= \delta_{a}^{c}\delta_{b}^{d} - \partial'_{a}\xi^{c}\delta_{b}^{d} - \delta_{a}^{c}\partial'_{b}\xi^{d} + \mathcal{O}(\xi^{2})$$

$$\Rightarrow g'_{ab}(x'^{i}) = (\delta_{a}^{c}\delta_{b}^{d} - \partial'_{a}\xi^{c}\delta_{b}^{d} - \delta_{a}^{c}\partial'_{b}\xi^{d})g_{cd} + \mathcal{O}(\xi^{2})$$

$$= g_{ab} - \partial'_{a}\xi^{c}g_{cb} - \partial'_{b}\xi^{d}g_{ad} + \mathcal{O}(\xi^{2})$$

Let us assume that ξ is/generates a local isometery. Then $g'_{ab} = g_{ab}$

Thus

$$g_{ab}(x^{c}) - g'_{ab}(x^{c}) =$$

$$(4.5) \overset{\text{Taylor}}{\Longleftrightarrow} g'_{ab}(x^{c}) = g'_{ab}(x'^{c}) - \xi^{c}\partial_{c}g'_{ab}(x^{c})$$

$$(4.9) \overset{\text{Tensor}}{\Longleftrightarrow} g_{ab}(x^{i}) = g'_{ab}(x'^{i}) + (\partial'_{b}\xi^{d}g_{ad} + \partial'_{a}\xi^{c}g_{cb})$$

From notes

$$\Rightarrow g_{ab}(X^{c}) - g'_{ab}(X^{c})$$

$$= \cancel{g'_{ab}(X'^{c})} + (\partial'_b \xi^d g_{ad} + \partial'_a \xi^c g_{cb})$$

$$- \cancel{g'_{ab}(X'^{c})} + \xi^c \partial_c g'_{ab}$$

$$= \xi^c \partial_c g'_{ab} + \partial'_b \xi^d g_{ad} + \partial'_a \xi^c g_{cb}$$

$$= \xi^c \partial_c g_{ab} + \partial_b \xi^d g_{ad} + \partial_a \xi^c g_{cb} + \mathcal{O}(\xi^2)$$

Recall $\frac{\partial X^c}{\partial X'^a} = \delta^c_a - \partial'_a \xi^c$

Therefore, the requirement $g_{ab} - g'_{ab} = 0$. is equivalent to

$$(\mathcal{L}_\xi g)_{ab} = 0 .$$

—/—

<u>4.3 Background and metric perturbation</u>

We will discuss the background equations (this is everything we have done up to this point) and add the perturbation correctly. Will work in conformal time because the equation tend to look neater. As before, we assume spacetime to be homogenous and isotrophic on large scales and derivation to be small. The metric will split into two points, the background and the perturbations.

For the background, we work with the FLRW meteric

$$ds^2 = {}^{(0)}g_{\alpha\beta}\, dX^\alpha dX^\beta$$
$$= -dt^2 + a^2(t)\, \gamma_{ij}\, dX^i dX^j$$
$$= a^2(\eta)\left[-d\eta^2 + \gamma_{ij}\, dX^i dX^j\right]$$

η is conformal time and γ_{ij} is the spatial part of the meteric

$$\gamma_{ij} = \delta_{ij} \frac{1}{\left(1 + \frac{k}{4}(x^2+y^2+z^2)\right)}$$

The background field equation are:

$$a'^2 + ka^2 = \frac{8\pi}{3} T^\eta{}_\eta\, a^4$$

$\leftarrow$ NOT an index

$$a'' + ka = \frac{4\pi}{3} T a^3$$

where $' = \frac{\partial}{\partial\eta}$

and T is the trace of the energy momentum tenser $T = T_\alpha^\alpha$. As before, the equations imply the conservation equation which becomes.

$$\frac{dT^\eta_\eta}{d\ln a} + (4T^\eta_\eta - T) = 0.$$

Perturbations.

Now we include perturbation, we write

$$ds^2 = {}^{(0)}g_{\alpha\beta}\, dX^\alpha dX^\beta + \delta g_{\alpha\beta}\, dX^\alpha dX^\beta$$

$\delta g_{\alpha\beta}$ is the perturbation and the complete meteric is

$$g_{\alpha\beta} = {}^{(0)}g_{\alpha\beta} + \delta g_{\alpha\beta}.$$

The perturbation $\delta g_{\alpha\beta}$ can be classified with respect to their transformation properties under purely spatial coordinate transformation on constant time hypersurfaces. These perturbations are of 3 types: scalar, vector, tensor.

In general: vector perturbation tend to decay in an expanding universe.

: tensor perturbations yield gravitation waves these waves do not couple to the energy and pressure inhomogenities.

It is the scalar perturbations which couple to the energy density and pressure. They may yield growing inhomogeneities which is imporant in the context of structure formation.

We will write:

$$\delta g_{\alpha\beta} = \begin{pmatrix} \delta g_{\eta\eta} & \delta g_{\eta j} \\ \delta g_{i\eta} & \delta g_{ij} \end{pmatrix}$$

The term $\delta g_{\eta\eta}$ is a scalar. We can view $\delta g_{\eta j}$ as a vector. However, we can write $\delta g_{\eta j} = \nabla_j f + V_j$ this we can decomposes $\delta g_{\eta j}$ as a vector and a scalar. How about δg_{ij}?

$$\delta g_{ij} = \nabla_i \nabla_j \tilde{f} + \frac{1}{2}(\nabla_i W_j + \nabla_j W_i) + t_{ij}$$

We have:

scalar pert. $\delta g_{\eta\eta}, \delta g_{\eta i}, \delta g_{ij}$
vector pert. $\delta g_{\eta i}, \delta g_{ij}$
tensor pert. δg_{ij}

Scalar perturbations

In general, these are written as follows:

$$\delta g^{(s)}_{\alpha\beta} = a^2(\eta) \begin{pmatrix} 2\phi & -\nabla_i B \\ -\nabla_i B & 2(\psi \gamma_{ij} - \nabla_{ij} E) \end{pmatrix}$$

$$\nabla_{ij} E = \nabla_i \nabla_j E$$

where ϕ, ψ, E and B are functions of space and time. The line element with scalar perturbation reads.

$$ds^2 = a^2(\eta) \Big[(1+2\phi) d\eta^2 - 2\nabla_i B \, dx^i d\eta$$

$$- \left((1-2\psi)\gamma_{ij} + 2\nabla_i \nabla_j E \right) dx^i dx^j \Big]$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega^2$$

$$\simeq -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$$

$$M << 1$$

Vector pert.

The most general form is:

$$\delta g^{(v)}_{\alpha\beta} = -a^2(\eta) \begin{pmatrix} 0 & -S_i \\ -S_i & \nabla_i F_j + \nabla_j F_i \end{pmatrix}$$

where F_i and S_i' are divergence-less vector fields $\nabla^i F_i = \nabla^i S_i' = 0$.

Tensor pert.

$$\delta g^{(T)}_{\alpha\beta} = -a^2(\eta)\begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}$$

where $h^i{}_i = 0$ and
$\nabla^i h^j{}_i = 0$.

Remark: The metric has 10 degrees of freedom, let's check if this consistent with our decompostion

tensor	$6-1-2$	$= 2$
vector	$6-1-1$	$= 4$
scalar		$= 4$
meteric.		$= 10$

4.4 Gauge invariables.

Consider the following infinitesimal coordinate transformation which preserves the scalar nature of the meteric pert.

$$\eta \mapsto \tilde{\eta} = \eta + \xi(\eta, x^i)$$

$$x^i \mapsto \tilde{x}^i + \delta^{ij}\nabla_j \xi(\eta, x^i)$$

From Section 4.1 we know how the metric changes under infinitesimal coordinate transformation. In lowest in ξ, we have

$$\delta g_{ab} - \delta \tilde{g}_{ab} = (\mathcal{L}_\xi g)_{ab}$$

$$\Rightarrow \delta \tilde{g}_{ab} = \delta g_{ab} - (\mathcal{L}_\xi g)_{ab}$$

A length calculation gives that the new function $\tilde{\Phi}$, $\tilde{\Psi}$ etc are related to ϕ, ψ, etc by.

$$\tilde{\phi} = \phi - \frac{a'}{a}\xi^0 - \xi^{0'}$$

$$\tilde{\psi} = \psi + \frac{a'}{a}\xi^0$$

$$\tilde{B} = B + \xi^0 - \xi'$$

$$\tilde{E} = E - \xi.$$

By combining the quantities with $\sim$ in a clever way, we can define new variable which will be invariant under the above coordinate transformation. The simplest such choice is

$$\Psi = \psi - \frac{a'}{a}(B - E')$$

and

$$\Phi = \phi + \frac{1}{a}\left[(B - E')a\right]'$$

In conformal Newton gauge (longitudinal gauge) one chooses $E = B = 0$. Then the metric perturbation are uniquely characterised by the two function Ψ and Φ. In the gauge we have

$$ds^2 = a^2(\eta)\left[(1+2\Phi)d\eta^2 - (1-2\Psi)\delta_{ij}\,dx^i dx^j\right]$$

Note that even a scalar $q(\eta, x^a)$ is not gauge invarient.

Let us consider

$$q(\eta, x^i) = q_0(\eta) + \delta q(\eta, x^i)$$

We have

$$\delta q - \delta\tilde{q} = (L_\xi q_0)$$

$$\Leftrightarrow \delta\tilde{q} = \delta q - (L_\xi q_0) = \delta q - \xi^\alpha \partial_\alpha q_0.$$

$$= \delta q - \xi^0 \delta q'.$$

Using the equatioi for $\tilde{B}$ and $\tilde{E}$ we can construct a gurge invariant scalar perturbation by noting

$$\tilde{B} - \tilde{E}' = B - E' + \xi^0$$

$$\Rightarrow \delta q^{(gi)} \overset{\text{gurge invarient}}{=} \delta q + (B - E')q_0'$$

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Question 1.

$(\mathcal{L}_\xi g)_{ab}$ ← Rank 2 tensor.

$$= \begin{pmatrix} \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \end{pmatrix}$$

Question 2, Need to prove that:

$$(\mathcal{L}_\xi g)_{ab} = 0$$

$$\delta q^{(gi)} = \delta q + (B - E')q_0'$$

to see this

$$\widetilde{\delta q}^{(gi)} = \widetilde{\delta q} + (\tilde{B} - \tilde{E}')\tilde{q}_0'$$

$$= \delta q - \xi^0 q_0' + (B - E' + \xi^0)q_0'.$$

$$= \delta q + (B - E')q_0' = \delta q^{(gi)}$$

Remark: Gauge here refers to coordinate transformations. It is called a gauge as we want perturbation that are physical and not just a consequence of a particular choice of coordinates. This effectivity reduces the number of degrees of freedom of scalar perturbatcoi to 2.

4.5 Perturbed field equatcois.

We are now interested in the perturbed (to first order). Einstein field equatcons. We also have to discuss matter perturbatcois.

4.5.1 Perturbed Einstein tensor components.

We now have our perturbed meteric.

$$g_{\mu\nu} \rightarrow \Gamma^{\lambda}_{\mu\nu} \longrightarrow R_{\mu\nu\alpha}{}^{\beta} \ldots$$

$$\ldots \rightarrow R_{\mu\nu} \rightarrow G_{\mu\nu}.$$

Example: Let us compute $\Gamma^{\eta}_{\eta\eta}$

Our meteric with $k=0$.

$$ds^2 = a^2(\eta)\left[(1+2\Phi)\,d\eta^2 - (1-2\Psi)\,\gamma_{ij}\,dx^i dx^j\right]$$

$k=0$, $\gamma_{ij}\,dx^i dx^j = dx^2 + dy^2 + dz^2$.

$$\Phi = \Phi(\eta, x, y, z)$$

$$\Psi = \Psi(\eta, x, y, z)$$

$$\Gamma^{a}_{bc} = \frac{1}{2} g^{ad}\left(g_{db,c} + g_{cd,b} - g_{bc,d}\right)$$

$$\Gamma^{\eta}_{\eta\eta} = \frac{1}{2} g^{\eta\alpha} \left(g_{\alpha\eta,\eta} + g_{\eta\alpha,\eta} - g_{\eta\eta,\alpha} \right).$$

$$= \frac{1}{2} g^{\eta\eta} g_{\eta\eta,\eta}$$

$$= \frac{1}{2} a^{-2}(\eta)(1+2\Phi)^{-1} \left[2aa'(1+2\Phi) + a^2 2\Phi' \right]$$

$$= \frac{1}{2}(1+2\Phi)^{-1} \left[2\frac{a'}{a} + 4\frac{a'}{a}\Phi + 2\Phi' \right]$$

$$\simeq \frac{1}{2}(1+2\Phi) \left[\frac{2a'}{a} + 4\frac{a'}{a}\Phi + 2\Phi' \right]$$

$$= \frac{a'}{a} + 2\frac{a'}{a} + \Phi' - 2\frac{a'}{a}\Phi$$

$$= \frac{a'}{a} + \Phi'.$$

Similar by $(1+2\Phi)$ to be $(1+2\varepsilon\Phi)$ $\varepsilon \ll 1$.

For the background equations

$${}^{(0)}G^{\eta}_{\eta} = \frac{3a'^2}{a^4}$$

$${}^{(0)}G^{i}_{\eta} = 0 \quad , \quad {}^{(0)}G^{i}_{j} = 0 \quad i \neq j$$

$${}^{(0)}G^{i}_{j} = \frac{(2a a'' - a'^2)}{a^4} \delta^{i}_{j}$$

The first order perturbed equations are

$$\delta G^{\eta}_{\eta} = 2\left[-3\frac{a'}{a}\left(\Psi' + \frac{a'}{a}\Phi\right) + \Delta\Psi\right] / a^2$$

$$\Delta\Psi = \partial_{xx}\Psi + \partial_{yy}\Psi + \partial_{zz}\Psi .$$

$$\delta G^{\eta}_{i} = 2\partial_i\left[\frac{a'}{a}\Phi + \Psi'\right] / a^2$$

$$\delta G^{j}_{i} = -2\Big[\Big\{\Psi'' + \frac{a'}{a}\left(\Phi' + 2\Psi'\right)$$

$$+ 2\frac{a''}{a}\Phi - \frac{a'^2}{a^2}\Phi\Big\}\delta^{i}_{j}$$

$$+ \left\{\frac{1}{2}\Delta(\Phi - \Psi)\right\}\delta^{i}_{j}$$

$$- \frac{1}{2}\partial_i\partial^i(\Phi - \Psi)\Big] / a^2$$

In particular:

$$\delta G^{j}{}_{i} = \partial_i\partial^j(\Phi - \Psi) / a^2 \qquad i \neq j .$$

<u>4.5.2 Matter perturbations.</u>

Our background energy-momentum tensor is

$$T^{\beta}_{\alpha} = (\rho + P)u_\alpha u^\beta - P\delta^{\beta}_{\alpha} .$$

The first order fluid perturbatcon are

$$\delta T_\alpha^{\ \beta} = \begin{pmatrix} \delta\rho & -({}^{(0)}\rho + {}^{(0)}P)\,\partial^i V/a \\ ({}^{(0)}\rho + {}^{(0)}P)\,\partial_i V_a & -\delta P \delta_i^{\ j} + \nabla_i^{\ j}\sigma \end{pmatrix}$$

$\delta\rho, \delta P$ perturbed energy density and pressure.

V velocity pontential.

σ anisotropic stress.

In the absence of anisotropic stress it follows that the spatial of the matter tensor is diagonal. Thus:

$$\delta G^i_{\ j} = 8\pi\, \delta T^i_{\ j} = 0 \quad i \neq j$$

$$\Rightarrow \partial_i \partial^i (\Phi - \Psi)/a^2 = 0.$$

$$\Rightarrow \boxed{\Phi = \Psi}.$$

4.5.3 Scalar field perturbations.

Much eaiser. We write

$$\varphi = \varphi_0(\eta) + \delta\varphi(\eta, x, y, z)$$

Putting this into the definition of the scalar field energy – momentum tensor gives.

$$^{(0)}T^{\eta}_{\eta} = \frac{1}{2a^2}\varphi_0'^2 + V(\varphi_0) = {}^{(0)}\rho_\varphi$$

$$^{(0)}T^{i}_{\eta} = 0.$$

$$^{(0)}T^{i}_{j} = \frac{1}{2a^2}\varphi_0'^2 - V(\varphi_0) = {}^{(0)}P_\varphi$$

and

$$\delta T^{\eta}_{\eta} = \left[-\varphi_0'^2\Phi + \varphi_0'\delta\varphi' + V_\varphi a^2 \delta\varphi\right] / a^2$$

$$\delta T^{\eta}_{i} = \varphi_0' \partial_i \delta\varphi / a^2$$

$$\delta T^{j}_{i} = \left[\varphi_0'^2\Phi - \varphi_0'\delta\varphi' + V_\varphi a^2 \delta\varphi\right]\delta^j_i$$

where

$$V_\varphi = \frac{dV}{d\varphi}.$$

We can also compute $\nabla_\alpha T^\alpha_\beta = 0$.

This gives ($\beta = \eta$ otherwise you get 0)

$$\delta\varphi'' + 2aH\delta\varphi' + k^2\delta\varphi + V_{\varphi\varphi}a^2\delta_\varphi = 0.$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$.

Typically for inflation $V_{\varphi\varphi}$ is of the order of the slowroll parameter and thus we can neglect it.

Hence, we will work with

$$\delta\varphi'' + 2aH\delta\varphi'' + k^2\delta_\varphi = 0.$$

<u>Example</u> : Perturbation in the matter dominated universe. (hardly asked in detial)

Let us consider the perturbation in the matter dominated universe. In conformal time the background evolve as.

$$a(\eta) = a_0\eta^2$$

$$^{(0)}\rho(\eta) = \frac{\rho_0}{a^3} \quad \text{and} \quad ^{(0)}P = 0.$$

Next we assume, no anisotropic stress and also dust-like perturbations this means $\delta P = 0$.

With $\Phi = \Psi$ the spatial part of the field equation is

$$\Psi'' + \frac{3a'}{a}\Psi' + \frac{2a''}{a}\Psi - \frac{a'^2}{a^2}\Psi = 0.$$

$a(\eta) = a_0\eta^2 \quad a'(\eta) = 2a_0\eta \quad a'' = 2a_0.$

$$\Psi'' + 3\frac{2}{\eta}\Psi' + 2\cdot\frac{2}{\eta^2}\Psi - \frac{4}{\eta^2}\Psi = 0.$$

$$\Rightarrow \Psi'' + \frac{6}{\eta}\Psi' = 0.$$

$f = \Psi' \Leftrightarrow f' + \frac{6}{\eta}f = 0.$ integrating factor with $C = C(x,y,z)$.

which is solved by

$$\Psi(\eta, x, y, z) = \Psi_0(x,y,z) + \frac{\Psi_1(x,y,z)}{\eta^5}$$

We are interested in the non-decaying part and so set $\Psi_1 = 0$.

Next, we consider the $(\eta\eta)$ equation

$$\delta G^{\eta}{}_{\eta} = 2\left[-3\frac{\grave{a}}{a}\left(\Psi' + \frac{\grave{a}}{a}\Phi\right) + \Delta\Psi\right] / a^2 = 8\pi\delta\rho .$$

$$\Psi = \Phi = \Psi_0(x, y, z)$$

$$\Rightarrow 2\left[-3\frac{2}{\eta}\left(0 + \frac{2}{\eta}\Psi_0\right) + \Delta\Psi_0\right] / a_0^2\eta^4 = 8\pi\delta\rho$$

$$\Leftrightarrow \frac{\left(-12\frac{1}{\eta^2}\Psi_0 + \Delta\Psi_0\right)}{a_0^2\eta^4} = 4\pi\delta\rho$$

We know that ${}^{(0)}\rho = \rho_0 / a^3 = \rho_0 / a_0^3\eta^6$.

We now define the so-called density contrast

$$\delta = \frac{\delta\rho}{{}^{(0)}\rho}$$

$$\Rightarrow \delta = \frac{\left(-12\frac{1}{\eta^2}\Psi_0 + \Delta\Psi_0\right)a_0^3\eta^6}{4\pi a_0^2\eta^4\rho_0} .$$

$$\ldots = \frac{a_0}{4\pi\rho_0}\left[-12\Psi_0 + \Delta\Psi_0\eta^2\right]$$

$$= \frac{a_0}{4\pi\rho_0}\left(-12\Psi_0 - k^2\Psi_0\eta^2\right)$$

$$= \frac{a_0}{4\pi\rho_0}(-)\left(12 + k^2\eta^2\right)\Psi_0.$$

Recall that the scalar factor is $a(\eta) = a_0\eta^2$. Hence, perturbations grows as the universe expands.

For a "proper" growth of perturbation, we would have expected exponenial growth. There is some structure formation but not enough.

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FIN

[signature]